Mathematics Exercise Sheet 1

1. Given that
$$cx = \sqrt{\left(\frac{ax^2 - b}{d}\right)}$$
express x in terms of a, b, c, and d.

2. If $x = p + \sqrt{q}$ where p and q are rational, show that x^2 and x^3 are of the form $P + Q\sqrt{q}$, where P and Q are rational.

3. Verify that $z = (4 + \sqrt{15})^{\frac{1}{3}} + (4 - \sqrt{15})^{\frac{1}{3}}$ satisfies
$$z^3 - 3z - 8 = 0.$$

4. Expand $(2 - 3x)^5$, arranging your answer in ascending powers of x with integer coefficients.

5. Given that $(1 + 2x)^{22} = 1 + Ax + Bx^2 + Cx^3 + \cdots$, find the values of A, B, and C.

6. Show that
$$\left(x + \frac{1}{x}\right)^3 + \left(x - \frac{1}{x}\right)^3 = 2x^3 + \frac{6}{x}.$$

7. Calculate the value of the term independent of x in the expansion of $\left(x^2 - \frac{3}{x}\right)^6$.

8. Show that
$$\frac{(2n)!}{n!} = 2^n.1.3.5.\cdots.(2n-1).$$

9. $(*)$ Write down the general term in the expansion of $(1+x)^n$. Use the identity
$$(1+x)^m(1+x)^n = (1+x)^{n+m}$$
to prove that
$${}_mC_r + {}_mC_{r-1}\cdot{}_nC_1 + {}_mC_{r-2}\cdot{}_nC_2 + \cdots + {}_nC_r = {}_{n+m}C_r.$$

10. Solve the equation $\frac{1}{x} + \frac{1}{3x-2} = 2$.

11. Given that α and β are roots of the equation $x^2 + 3x - 6 = 0$, find a quadratic equation with integer coefficients whose roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$.

12. Find the set of values of k for which the equation $x^2 + kx + (3 - k) = 0$ has real roots. In the case when $k = 5$, the roots of the equation are α and β. *Without calculating the values of α and β*, find

   (a) the value of $\alpha^3 + \beta^3$;

   (b) a quadratic with roots $\alpha^2 + 3\beta$ and $\beta^2 + 3\alpha$.

13. Divide $x^6 + 5x^5 + 11x^4 + 13x^3 - 3x^2 - 8x + 5$ by $x^2 + 2x + 5$.

14. Show that $x - 4$ is a factor of $f(x) = x^3 - 8x^2 + 29x - 52$. Factorise $f(x)$ and show that the equation $f(x) = 0$ has only one real root.

15. Use the remainder theorem to find a factor of $f(x) = 2x^3 - 9x^2 + 7x + 6$, and hence factorise $f(x)$ into its linear factors.

16. The function $f(x)$ is given by $f(x) = x^3 + ax^2 - 4x + b$, where a and b are constants. Given that $x - 2$ is a factor of $f(x)$ and that there is a remainder of 6 when $f(x)$ is divided by $x + 1$, find the values of a and b.

17. Show that
$$\frac{1}{1+x} - \frac{8}{2-x} + \frac{12}{(2-x)^2} = \frac{kx^2}{(1+x)(2-x)^2}$$
where k is an integer to be determined.

18. Express
$$\frac{1+3x^2}{(1+x)^2(1+3x)}$$
in partial fractions.

19. Express
$$\frac{1-2x+5x^2}{(1-2x)(1+x^2)}$$
in partial fractions.

20. $(*)$ Express $x^4 - 4x^2 + 16$ in the form
$$(x^2 + Ax + B)(x^2 + Cx + D)$$
where A, B, C, and D, are real constants. Hence express
$$\frac{1}{x^4 - 4x^2 + 16}$$
in partial fractions.

21. $(*)$ Express
$$\frac{x^5 - 1}{x^2(x^3 + 1)}$$
in partial fractions.