

## Mathematics Exercise Sheet 4

1. Differentiate the following functions with respect to  $x$ :

- (a)  $4x^3 - 2x^2 + 1 + x^{-2}$ ,
- (b)  $\cos(3x + 2)$ ,
- (c)  $\frac{x}{x^2+1}$ ,
- (d)  $x \ln x$ ,
- (e)  $\sqrt{1+x^2}$ ,
- (f)  $e^x \cos(x^2)$ ,
- (g)  $\frac{x-8}{(x+2)(2x-1)}$ ,
- (h)  $x^{e^x}$ ,
- (i)  $\cos(\tan(x^2))$ ,

2. Differentiate the following functions with respect to  $x$ :

- (a)  $x^3 + 3xy^2 - 2xy + x^{-4}y^{-3} = 0$ ,
- (b)  $\cos(x) \sin(y) = 1$ ,
- (c)  $\cos(y \tan(x)) = \ln(xy)$ ,
- (d)  $\frac{x+2 \sin(y)+4}{(x-y)} = \cot(y)$ ,
- (e)  $y \tan(x^{-1}) + x \sec(y^{-1}) = e^x$ ,

3. Differentiate the following functions with respect to  $x$ :

- (a)  $x = t^2 + 2t + 1, y = 7t^3$ ,
- (b)  $x = \cos(t), y = \cot(t)$ ,
- (c)  $x = \ln(\sqrt{1+2t^2}), y = t^3 + t$ ,
- (d)  $x = \cos(t)e^{t^2}, y = \sin(t)e^{-t^2}$ ,
- (e)  $x = \frac{1}{\ln(t)}, y = 3 \tan(4t)$ .

4. Given that  $y = e^{-x} \sin(x\sqrt{3})$ , prove that

$$\frac{dy}{dx} = -2e^{-x} \sin\left(x\sqrt{3} - \frac{\pi}{3}\right).$$

Show also that  $\frac{d^3y}{dx^3} = ky$  for some constant  $k$ , and state the value of  $k$ .

5. If  $f(x) = 3 - \frac{x^2}{4} + \ln\left(\frac{x}{2}\right)$ ,

- (a) show that there is a root  $\alpha$  of  $f(x) = 0$  such that  $0.09 < \alpha < 0.1$
- (b) find  $f'(x)$  and obtain the value of  $\beta$  such that  $f'(\beta) = 0$ .

6. (\*) If  $y = \ln\left(1 + \sqrt{1+x}\right)$ , show that

$$4x \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} = (1+x)^{-\frac{3}{2}}.$$

7. Use Leibnitz' theorem to compute the fifth derivative of

$$(a) x^3 \cos x \quad (b) x^2 \ln x.$$

8. (\*) If  $y = \frac{x+1}{x^2+2x+3}$  show that

$$(x^2 + 2x + 3) \frac{d^2y}{dx^2} + 4(x+1) \frac{dy}{dx} + 2y = 0.$$

Use Leibnitz' theorem to establish that

$$(x^2 + 2x + 3) \frac{d^{n+2}y}{dx^{n+2}} + 2(n+2)(x+1) \frac{d^{n+1}y}{dx^{n+1}} + (n+1)(n+2) \frac{d^ny}{dx^n} = 0.$$