Mathematics Exercise Sheet 5

1. Find the gradient of the curve at the point for which $x = 4$ in each of the following cases:

   (a) $y = x^2 - 3\sqrt{x}$,

   (b) $y = \cos(\frac{1}{4}\pi x)$.

2. Find the equation of the tangent at the point $(1, 0)$ to the curve $y = (x+1)\ln x$, $x > 0$, and verify that it meets the line $x = 2$ at the point $(2, 2)$.

3. Given $y = e^{-x^2}$ find expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Hence find the x coordinates of the two points on the graph for which $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is equal to zero. Show that these are both points of inflection.

4. Find the maximum and minimum values of $4\sin x + \frac{9}{(1+\sin x)}$ for $0 \leq x \leq \pi$.

5. The function f is defined by $f(x) = 4x^2 - 3 - \frac{1}{x}$ with $x \neq 0$.

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Find the values of a such that $f'(a) = 0$, and calculate $f''(a)$ in these cases. What information does this give you about the graph of f?

   (c) Find the values of x where the graph $y = f(x)$ meets the x-axis.

   (d) Sketch the curve $y = f(x)$. Explain how the shape of the graph is related to the terms in the expression for $f(x)$: (i) when x is near 0 and (ii) when x is large (positive or negative).

6. Express $f(x) = \frac{(3x^2+1)^2}{x^2}$ in the form $Ax^2 + Bx + \frac{C}{x^2}$. Hence evaluate

$$\int_1^2 f(x)dx.$$

7. Differentiate $\ln(x^2 - 2x + 2)$ with respect to x. Hence find

$$\int_1^2 \frac{x-1}{x^2-2x+2}dx.$$

8. Evaluate the integral

$$\int_0^1 \frac{x^3}{1+x^4}dx.$$

9. Using the substitution $y = 2x - 1$, evaluate the integral

$$\int_1^2 \frac{x}{(2x-1)^3}dx.$$

10. Express $f(x) = \frac{4-6x^2}{(1+2x)^2(1+x)}$ in partial fractions. Show that

$$\int_0^1 f(x)dx = \frac{5}{3} + \ln\frac{\sqrt{3}}{4}.$$

11. Find

$$\int \sin^2 x \cos x dx.$$

12. Find

$$\int \frac{e^{3x} - e^{2x}}{e^x}dx.$$

13. Find

$$\int x\sqrt{4x-3}dx.$$

14. Find

$$\int \tan x \sec^2 x dx.$$

15. Find

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx.$$

16. Find

$$\int (x-5)^4(x+3)^2dx.$$

17. Find

$$\int \frac{\sin 3x}{1+\cos 3x}dx.$$

18. Find

$$\int 2\sin 7x \cos 3x dx.$$

19. Find

$$\int \cos^4 x dx.$$