

Mathematics Exercise Sheet 7

1. Determine the following limits:

(a)

$$\lim_{x \rightarrow -2} \frac{x^2 + 6x + 8}{x + 2}.$$

(b)

$$\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x^2 - 4}}.$$

(c)

$$\lim_{x \rightarrow 0} \frac{\tan x}{x}.$$

(d)

$$\lim_{x \rightarrow 0} \frac{1 - \cos(ax)}{x^2}.$$

(e)

$$\lim_{x \rightarrow 0} \frac{(x^2 + 2) \sin(3x)}{x}.$$

(f)

$$\lim_{x \rightarrow \infty} \frac{x^2 + x + 1}{x^2 - x + 1}.$$

(g) (*)

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x + \sqrt{x+1}}}{\sqrt{x + \sqrt{1 + \sqrt{x}}}}.$$

2. Verify the Maclaurin series given in lectures for $\sin(x)$, $\cos(x)$, $\ln(1+x)$.

3. (*) Give the n th term of the Maclaurin series for each of the following functions:

(a) $2 \sin x \cos x$.

(b) $\ln \sqrt{\frac{1-x}{1+x}}$.

(c) $\ln[(1+x)(1-2x)^2]$.

(d) $\sin^3 x$.

4. Give the Taylor series for $\cos x$ about $\frac{\pi}{2}$ to three terms.

5. If $y = \tan(e^x - 1)$, prove that

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(1 + 2e^xy).$$

Obtain the Maclaurin expansion of y up to and including the x^4 term.

6. For each of the following sets determine (i) the elements of the set and (ii) the subsets of the set.

(a) $A = \{1, 2, 3\}$

(b) $B = \{R, \{G\}, B\}$

(c) $C = \{R, \{R\}\}$

7. Determine whether each of the following statements is true or false.

(a) $2 \in \{1, 2, 3\}$.

(b) $2 \subset \{1, 2, 3\}$.

(c) $\{2\} \in \{1, 2, 3\}$.

(d) $\{2\} \subseteq \{1, 2, 3\}$.

(e) $\emptyset \in \{\emptyset\}$.

(f) $\emptyset = \{\emptyset\}$.

8. For each of the following statements, determine (with reasons) whether it is true or false. For each false statement, give an example to show that it fails.

(a) If $A \subset C$ and $B \subseteq C$ then $A \subset B$.

(b) If $A \subset B$ and $B \subset C$ then $A \subseteq C$.

(c) If $A \subseteq B$ and $B \subseteq C$ then $A \subset C$.

(d) If $A \subseteq B$ and $A \cap B \neq \emptyset$ then $A \subset B$.

(e) If $A \subseteq B$ and $A \cup B \neq A$ then $A \subset B$.

Answers

1. (a) 2, (b) 0, (c) 1, (d) $\frac{a^2}{2}$, (e) 6, (f) 1, (g) 1.

3. (a) $\frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$, (b) $\frac{-x^{2n+1}}{2n+1}$,
 (c) $\frac{x^n}{n} [(-1)^{n+1} - 2^{n+1}]$, (d) $\frac{(-1)^n}{4(2n+1)!} (3 - 3^{2n+1}) x^{2n+1}$.

4. $-(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^3 - \frac{1}{120}(x - \frac{\pi}{2})^5$.

5. $x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{13}{24}x^4$.

6. (a) (i) 1, 2, 3. (ii) $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.

(b) (i) $R, \{G\}, B$. (ii) $\emptyset, \{R\}, \{\{G\}\}, \{B\}, \{R, \{G\}\}, \{R, B\}, \{\{G\}, B\}, \{R, \{G\}, B\}$.

(c) (i) $R, \{R\}$. (ii) $\emptyset, \{R\}, \{\{R\}\}, \{R, \{R\}\}$.

7. (a) T , (b) F , (c) F , (d) T , (e) T , (f) F .

8. (a) F , e.g. $A = \{1\}, B = \{2\}, C = \{1, 2\}$, (b) T ,

(c) F , could have $A = B = C$, (d) F , could have $A = B$, (e) T .