

# MA1601: Mathematics

## 0. Introduction

The aim of this course is to review the basic mathematics which you have already learnt during A-level. You should find it almost entirely familiar, with only the occasional topic of new material.

However, even for those parts which are familiar there will be aspects which will be treated differently at university level. Most importantly, you will be expected to *know* key facts and formulas — you should not expect to have a formula sheet at your disposal. Thus, for example, you will be expected to memorise all the standard integrals and derivatives, the trigonometric identities, etc.

There are several books recommended on the course webpage. My personal recommendation would be the A-level textbook **Bostock and Chandler: Pure Mathematics** (possibly together with **Further Pure Mathematics** by the same authors with Rourke).

An alternative would be **Jordan and Smith: Mathematical Techniques**. This goes more quickly through the basic material in this course, but goes on to cover more advanced topics that you will see in many of your first and second year modules.

The **integers**  $\mathbb{Z}$  consist of all whole numbers  $0, \pm 1, \pm 2, \dots$

An integer  $a$  is **divisible** by another (non-zero) integer  $b$  if there exists a third integer  $c$  such that  $a = bc$ . In this case we call  $b$  a **divisor** of  $a$ . An integer  $p$  is **prime** if  $p > 1$  and  $p$  has no positive divisors except 1 and  $p$ .

Although easy to define, integers are hard to completely understand. For example, we do not have a formula for determining quickly whether a given number is prime.

Primes are important because of

The **rational numbers**,  $\mathbb{Q}$ , consist of all numbers of the form  $r = \frac{p}{q}$  where  $p$  and  $q$  are integers with  $q \neq 0$ . Note that there are equivalent forms of a rational number:

$$\frac{p}{q} = \frac{s}{t} \quad \text{if and only if} \quad pt = qs.$$

**Example 1.1.4:**  $\frac{3}{5} = \frac{9}{15}$  as  $3 \times 15 = 5 \times 9$ .

We usually simplify fractions to the form  $r = \frac{p}{q}$  where  $\text{hcf}(p, q) = 1$ . (Integers with  $\text{hcf}(p, q) = 1$  are called **coprime**.)

You will also notice that the pace of university mathematics is much faster than at school. This will certainly be true of the revision of A-level material, but will also extend to other courses. In part this will be because there will be fewer worked examples; you will be expected to practise calculations by yourself. Also, if you do not keep up to date, the speed of the course will make it hard for you to catch up.

In all courses it is important that you attempt the exercise sheets. These will not be marked, but without working through them you are very unlikely to perform well in the final exams. Tutors are very pleased when students ask questions about material they do not understand — you should make full use of them!

## 1. Arithmetic

In this chapter we will review the basic algebraic manipulations which should already be familiar. First we introduce the main classes of numbers.

### 1.1 Numbers

Most basic are the **natural numbers**,  $\mathbb{N}$ , which consist of the positive whole numbers  $1, 2, 3, \dots$  (Some textbooks include 0 as a natural number.) Note in passing that **positive** means  $> 0$ , and **negative** means  $< 0$ . To talk about numbers  $\geq 0$  we say **non-negative**.

**Theorem 1.1.1: (The fundamental theorem of arithmetic)**  
*Every positive integer has a unique prime factorisation.*

Note that this result says two things: there **is** a factorisation as a product of primes, and it is **unique**.

**Example 1.1.2:**  $2,522,520 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 7 \times 11 \times 13$ .

Given two non-zero integers  $m$  and  $n$  we define their **highest common factor**  $\text{hcf}(m, n)$  to be the largest divisor of  $m$  and  $n$ , and the **least common multiple**  $\text{lcm}(m, n)$  to be the smallest positive integer divisible by  $m$  and  $n$ .

**Example 1.1.3:** If  $m = 60 = 2 \times 2 \times 3 \times 5$  and  $n = 70 = 2 \times 5 \times 7$ , then  $\text{hcf}(m, n) = 10$  and  $\text{lcm}(m, n) = 420$ .

If we imagine numbers as making up a line, then in any given segment there are infinitely many rationals. However, not every number is rational. For example,  $\sqrt{2}$  is **not** rational (this will be proved in the Mathematical Communication course). We will call such numbers **irrational**.

The **real numbers**,  $\mathbb{R}$ , consist of all rational and irrational numbers.

Note that we have not given a precise definition of  $\mathbb{R}$ , as we have not really said what irrational numbers are. This is because  $\mathbb{R}$  is rather hard to define! It took most of the nineteenth century for mathematicians to come up with a definition which actually reflected the properties of real numbers that we 'know' that we require.

**Remark 1.1.5:** **Never** approximate fractions, square roots, etc., by decimals, unless you are **specifically** asked for an approximate answer.



In general the number of permutations of  $r$  objects from a set of  $n$  distinct objects is given by

$${}_nP_r = \frac{n!}{(n-r)!}$$

and the number of combinations is just  ${}_nC_r = \frac{{}_nP_r}{r!}$ , which equals

$${}_nC_r = \frac{n!}{r!(n-r)!}.$$

Note that this is the same as the coefficient in the binomial theorem.

## 1.5 Polynomials

A **polynomial of degree  $n$**  in  $x$  is a function  $p(x)$  of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where  $a_0, a_1, \dots, a_n$  are constants with  $a_n \neq 0$ .

We call degree 2 polynomials **quadratic**, degree 3 **cubic** etc.

Traditionally we write quadratics in the form

$$ax^2 + bx + c.$$

To **complete the square** we write a quadratic in the form

$$a\left(x + d\right)^2 + e$$

for some constants  $a, d$ , and  $e$ . In this case the roots (if they exist) are given by

$$x = -d \pm \sqrt{-e}$$

and if  $a$  is positive (respectively negative) then the minimum (respectively maximum) occurs at  $x = -d$ , and equals  $ae$ .

It is well known that the roots of  $ax^2 + bx + c$  are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So we have

- two distinct roots if  $b^2 - 4ac > 0$ ,
- one root if  $b^2 - 4ac = 0$ ,
- no roots if  $b^2 - 4ac < 0$ .

**Example 1.5.2:** If the roots of  $x^2 + 5x + 3 = 0$  are  $\alpha$  and  $\beta$ , find the quadratic equation with roots  $\alpha^3$  and  $\beta^3$ .

We have  $\alpha + \beta = -5$  and  $\alpha\beta = 3$ . So

$$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 \\ &= -125 - 3\alpha\beta(\alpha + \beta) \\ &= -125 - 9(-5) = -80. \end{aligned}$$

and  $\alpha^3\beta^3 = (\alpha\beta)^3 = 27$ . Thus the required equation is

$$x^2 + 80x + 27 = 0.$$

**Example 1.5.1:** We will complete the square for the following quadratic.

$$\begin{aligned} f(x) &= 3x^2 + 2x - 4 \\ &= 3\left(x^2 + \frac{2x}{3} - \frac{4}{3}\right) \\ &= 3\left(x + \frac{1}{3}\right)^2 - \frac{13}{3} \end{aligned}$$

has roots  $x = -\frac{1}{3} \pm \sqrt{\frac{13}{9}}$  and a minimum at  $x = -\frac{1}{3}$  of  $-\frac{13}{3}$ .

If we denote the roots by  $\alpha$  and  $\beta$  then we have

$$f(x) = a(x - \alpha)(x - \beta) = ax^2 + bx + c$$

and so

$$a(x^2 - (\alpha + \beta)x + \alpha\beta) = ax^2 + bx + c.$$

From this we deduce that

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

Similar formulas can be deduced for cubics, quartics, etc.

Returning to general polynomials, we can easily add and multiply them to form new polynomials. However,  $p(x)/q(x)$  is not in general a polynomial.

**Example 1.5.3:** Let  $p(x) = x^2 + 1$  and  $q(x) = x - 2$ . Then

$$p(x) + q(x) = x^2 + x - 1$$

and

$$p(x)q(x) = (x^2 + 1)(x - 2) = x^3 - 2x^2 + x - 2.$$

For  $p(x)/q(x)$  we have

$$\begin{array}{r} x + 2 \\ x - 2 \overline{) x^2 + 0x + 1} \\ \underline{x^2 - 2x} \phantom{+ 1} \\ 2x + 1 \\ \underline{2x - 4} \\ 5 \end{array}$$

and so  $p(x)/q(x)$  equals

$$\frac{x^2 + 1}{x - 2} = x + 2 + \frac{5}{x - 2}.$$

Let  $p(x)$  be a polynomial of degree  $n$  and divide  $p(x)$  by  $x - a$ , where  $a$  is a constant:

$$\frac{p(x)}{x - a} = q(x) + \frac{r}{x - a}$$

where  $q(x)$  is a polynomial and  $r$  is a constant, i.e.

$$p(x) = (x - a)q(x) + r.$$

From this we deduce

**Theorem 1.5.4:** If  $p(x)$  is a polynomial with  $p(a) = r$  then

$$p(x) = (x - a)q(x) + r$$

for some polynomial  $q(x)$ .

When  $r \neq 0$  this is called the **remainder theorem** and when  $r = 0$  it is called the **factor theorem**.

**Example 1.5.5:** Factorise

$$f(x) = x^3 - 7x^2 + 7x + 15.$$

We try some numbers:  $f(0) = 15$ ,  $f(1) = 16$ ,  $f(-1) = 0$ , and so  $x + 1$  is a factor.

$$\begin{aligned} f(x) &= (x + 1)(x^2 - 8x + 15) \\ &= (x + 1)(x - 3)(x - 5). \end{aligned}$$

Standard results (to be **memorised**):

$$\begin{aligned} x^2 - a^2 &= (x - a)(x + a) \\ x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\ x^n - a^n &= (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}) \end{aligned}$$

When  $n$  is odd we can get a formula for  $x^n + a^n$  from the last one by replacing  $a$  by  $(-a)$ . However, there is no simple formula for the case  $n$  even.

**Fact:** Every polynomial can be factorised into linear and/or quadratic terms.

If  $p(x) = a_nx^n + \dots + a_0$  has  $n$  distinct roots  $x_1, \dots, x_n$ , then

$$p(x) = a_n(x - x_1)(x - x_2) \dots (x - x_n).$$

### The method of undetermined coefficients

If two polynomials are identical — i.e. are equal for every value of  $x$  — then the coefficients of like terms are equal.

**Example 1.5.6:** Find  $a, b, c, d$  such that

$$r^3 = ar(r - 1)(r - 2) + br(r - 1) + cr + d.$$

Expanding we see that

$$\begin{aligned} r^3 &= a(r^3 - 3r^2 + 2r) + b(r^2 - r) + cr + d \\ &= ar^3 + (b - 3a)r^2 + (2a - b + c)r + d. \end{aligned}$$

Therefore  $a = 1$ ,  $b - 3a = 0$ ,  $2a - b + c = 0$  and  $d = 0$ ; i.e.  $a = 1$ ,  $b = 3$ ,  $c = 1$ , and  $d = 0$ .

## 1.6 Rational functions

A **rational function** is a function of the form  $\frac{p(x)}{q(x)}$  where  $p(x)$  and  $q(x)$  are polynomials with  $q(x)$  not identically zero. (That is, there is at least one value of  $x$  for which  $q(x) \neq 0$ .) For example

$$\frac{x^2 + 6x + 4}{x^2 - 5} \quad \text{and} \quad \frac{x + 7}{3x^7 - 2x + 1}.$$

We can add or subtract rational functions just like we do ordinary fractions. We can also simplify them in the same way (by removing common factors from the top and bottom).

A **proper** rational function is one where the degree of the numerator is less than the degree of the denominator. Otherwise we say the function is **improper**. For example, the first fraction above is improper, the second proper.

There is a second way to simplify a rational function which has a product of factors in the denominator, using partial fractions. To do this to a fraction  $p(x)/q(x)$  we use the following procedure:

**Step 1:** Simplify  $p/q$  to form a proper rational function.

**Step 2:** Factorise the denominator into linear and quadratic factors.

**Step 3:** If  $q$  has  $n$  factors, write the fraction as a sum of  $n$  terms using the following correspondence between factors of  $q$  and summands:

$$(x-a)^r \longleftrightarrow \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_r}{(x-a)^r}$$

$$(ax^2+bx+c)^r \longleftrightarrow \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

where  $A_i$  and  $B_i$  (with  $1 \leq i \leq r$ ) are constants.

The total number of constants equals the degree of the denominator. These constants can be determined by using the method of undetermined coefficients.

**Example 1.6.1:**

$$\frac{x+5}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}.$$

Therefore

$$x+5 = A(x+1) + B(x-3).$$

We could equate coefficients, instead we substitute values chosen to make most terms disappear. Substituting  $x = -1$  and  $x = 3$  we obtain

$$4 = -4B \quad \text{and} \quad 8 = 4A$$

i.e.  $A = 2$  and  $B = -1$ .

**Example 1.6.2:**

$$\frac{2x^2+x-2}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}.$$

Therefore

$$2x^2+x-2 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3.$$

Substituting  $x = 0$  and  $x = 1$  we obtain

$$-2 = -C \quad \text{and} \quad 1 = D$$

Comparing coefficients of the  $x^3$  terms and the  $x^2$  terms we obtain

$$0 = A + D \quad \text{and} \quad 2 = -A + B$$

and hence  $A = -1$ ,  $B = 1$ ,  $C = 2$ ,  $D = 1$ .

**Example 1.6.3:**

$$\frac{5x-12}{(x+2)(x^2-2x+3)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+3}$$

as  $x^2 - 2x + 3$  cannot be factorised. Therefore

$$5x-12 = A(x^2-2x+3) + (Bx+C)(x+2).$$

Substituting  $x = -2$  we obtain  $A = -2$ . By comparing coefficients of the  $x^2$  terms and constant terms we obtain  $B = 2$  and  $C = -3$ .

**Example 1.6.4:**

$$\begin{aligned} \frac{3x^3-x^2+2}{x(x^2-1)} &= \frac{3(x^3-x)-x^2+3x+2}{x(x-1)(x+1)} \\ &= 3 - \frac{(x^2-3x-2)}{x(x-1)(x+1)}. \end{aligned}$$

Now

$$\frac{(x^2-3x-2)}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

and we can show that  $A = 2$ ,  $B = -2$  and  $C = 1$ . Therefore

$$\frac{3x^3-x^2+2}{x(x^2-1)} = 3 - \frac{2}{x} + \frac{2}{x-1} - \frac{1}{x+1}.$$