1.7 Sums of series

We often want to sum a series of terms, for example when we look at polynomials. We abbreviate a sum of the form

$$u_1 + u_2 + \cdots + u_r \quad \text{by} \quad \sum_{i=1}^{r} u_i.$$

For example

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

and

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

Suppose that $u_i = a + (i-1)d$, so that u_i with $i \geq 1$ form an arithmetic progression (AP) with initial value a and common difference d. Then

$$\begin{aligned} \textstyle\sum_{i=1}^{n} u_i &= a + (a+d) + \cdots + (a + (n-1)d) \\ &= na + d + 2d + \cdots + (n-1)d \\ &= na + d\tfrac{n(n-1)}{2} = \tfrac{1}{2}n(2a + (n-1)d). \end{aligned}$$

Next suppose that $u_i = ar^{i-1}$, so that u_i with $i \geq 1$ form a geometric progression (GP) with initial value a and common ratio r. Then

$$\sum_{i=1}^{n} u_i = a + ar + \cdots + ar^{n-1} = \begin{cases} na & \text{if } r = 1 \\ \frac{a(1-r^n)}{1-r} & \text{if } r \neq 1. \end{cases}$$

(To verify the second case, rearrange the expression for $1^n - r^n$ given in Section 1.5.)

We can also sum certain series of powers of consecutive integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 1.7.1: The fourth term in a geometric progression is 7 and the seventh is 4. Find the sum S_{18} of the first eighteen terms.

We have $u_4 = ar^3 = 7$ and $u_7 = ar^6 = 4$. Therefore

$$\frac{ar^6}{ar^3} = \frac{4}{7} \quad \text{and so} \quad r = \left(\frac{4}{7}\right)^{\frac{1}{3}}.$$

Substituting into the expression for u_4 we deduce that $a = \frac{49}{4}$. Then

$$S_{18} = \left(\frac{49}{4}\right) \frac{1 - (\frac{4}{7})^{\frac{18}{3}}}{1 - (\frac{4}{7})^{\frac{1}{3}}} = \left(\frac{49}{4}\right) \frac{1 - (\frac{4}{7})^{6}}{1 - (\frac{4}{7})^{\frac{1}{3}}}.$$

Example 1.7.2: Find the sum S_n of the squares of the first n even integers greater than zero.

$$\begin{aligned} S_n &= 2^2 + 4^2 + \cdots + (2n)^2 \\ &= \sum_{k=1}^{n} (2k)^2 = \sum_{k=1}^{n} 4k^2 = 4\sum_{k=1}^{n} k^2 \\ &= \frac{4}{6} n(n+1)(2n+1). \end{aligned}$$

2. Real functions of one variable

2.1 General definitions

A real function is a rule that assigns to each real number in some set another real number, in a unique fashion. The set of inputs is called the domain of the function, and the set of outputs is called the range or image.

Usually we talk about a function going from one set to another without guaranteeing that every value in the latter set occurs as an output of the function. We refer to such a target set as the codomain. Thus the range is a subset of the codomain.

Let D_f be the domain of f, with codomain C_f and range R_f. We write this as

$$f : D_f \longrightarrow C_f \quad \text{or} \quad f : x \longmapsto f(x)$$

where $x \in D_f$ (and $f(x) \in C_f$). This has the advantage over the form $f(x) = \cdots$ that we do not need to give an explicit formula for f.

Example 2.1.1: Let $f(x) = x^2$ with $x \in \mathbb{R}$.

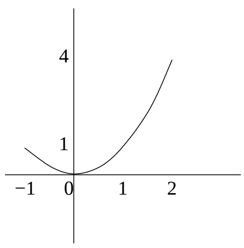
This has domain $\mathbb{R}$, i.e. $-\infty < x < \infty$, and range the set of y with $y \geq 0$.

Example 2.1.2: Take f as in the preceding example, but with $-1 \leq x \leq 2$.

This has domain $-1 \leq x \leq 2$ and range $0 \leq y \leq 4$.

The graph of a function is the set $\{(x, y) : y = f(x),\ x \in D_f\}$ which is a subset of the plane $\mathbb{R}^2$. We often represent this graphically.

Example 2.1.3: The graph for Example 2.1.2 is $\{(x, x^2) : -1 \leq x \leq 2\}$

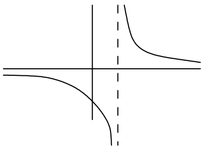


If the domain of a function is not specified, we assume that it is the largest set of real numbers on which the function is defined.

Example 2.1.4: Specify the domain and range of $f(x) = \frac{1}{x-2}$.

Domain: any real number except 2.
Range: Can we solve $y = \frac{1}{x-2}$?
No if $y = 0$. If $y \neq 0$ then

$$\frac{1}{y} = x - 2 \qquad \text{and} \qquad x = 2 + \frac{1}{y}.$$



Therefore the range is all real numbers except zero.

The composition of two functions f and g, written $f \circ g$, or just fg, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

This only makes sense if $g(x)$ is contained in the domain of f, so the domain of $f \circ g$ is the set of all $x \in D_g$ such that $g(x) \in D_f$.

Example 2.1.5: Let $f(x) = 3x^2 - 2x + x^{-1}$ with $x \neq 0$ and $g(x) = 2x + 1$ with $x \in \mathbb{R}$.

$$(f \circ g)(x) = f(2x+1) = 3(2x+1)^2 - 2(2x+1) + \frac{1}{2x+1}$$

which has domain $x \neq -\frac{1}{2}$.

$$(g \circ f)(x) = g(3x^2 - 2x + x^{-1}) = 2(3x^2 - 2x + x^{-1}) + 1$$

which has domain $x \neq 0$.

A function f is one-to-one (1–1) or injective if $x \neq y$ implies that $f(x) \neq f(y)$.

Example 2.1.6: $f(x) = x + 1$ with $x \in \mathbb{R}$ is injective as if $f(x) = f(y)$ then

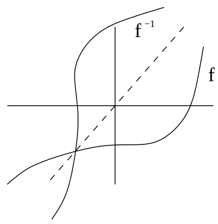
$$x + 1 = y + 1 \qquad \text{so} \qquad x = y.$$

$f(x) = x^2$ with $x \in \mathbb{R}$ is not injective, as $f(2) = f(-2)$.

An injective function f has an inverse f^{-1}. For each b in the image of f, we set $f^{-1}(b)$ to be the unique element a in the domain of f such that $f(a) = b$. So $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$. Also

$$f \circ f^{-1}(x) = x \qquad \text{and} \qquad f^{-1} \circ f(x) = x.$$

The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.



Example 2.1.7: Let $f(x) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}$ for $x \neq -1$.
Set $y = f(x)$, so

$$(x+1)y = x - 1.$$

Rearranging we get that

$$x = \frac{1+y}{1-y}$$

and hence $f^{-1}(x) = \frac{1+x}{1-x}$ with $x \neq 1$.

To check:

$$f \circ f^{-1}(x) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{1 + x - 1 + x}{1 + x + 1 - x} = \frac{2x}{2} = x.$$

Note that it is not possible to talk about the inverse of a non-injective function. For example, consider $f(x) = x^2$ with $x \in \mathbb{R}$. If $f^{-1}(4)$ exists, is it 2 or -2?

However, $f(x) = x^2$ with $x \geq 0$ does have an inverse: $f^{-1}(x) = \sqrt{x}$. This is one reason why we may restrict the domain of a function.

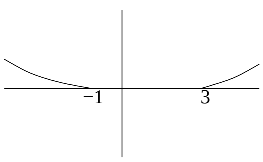
2.2 Special functions

We have already considered certain special classes of functions: polynomials, and rational functions. Here are a few more.

The square root function $f(x) = \sqrt{x}$ where $x \geq 0$. (Recall that we have already defined this function in Section 1.2.)

Example 2.2.1: Find the domain and range of $\sqrt{x^2 - 2x - 3}$.

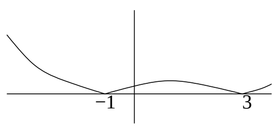
Set $y = h(x) = \sqrt{x^2 - 2x - 3} =$
$f \circ g(x)$ where
$g(x) = x^2 - 2x - 3$ and
$f(x) = \sqrt{x}$.
The domain is $x^2 - 2x - 3 \geq 0$,
i.e. $(x+1)(x-3) \geq 0$.
So $x \geq 3$ or $x \leq -1$.
The range is $y \geq 0$.



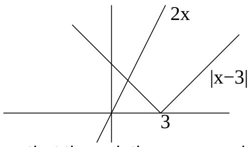
The modulus function $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

Example 2.2.2: Sketch the graph of $f(x) = |x^2 - 2x - 3|$.

$$f(x) = \begin{cases} x^2 - 2x - 3 & \text{if } x \leq -1 \\ -x^2 + 2x + 3 & \text{if } -1 < x < 3 \\ x^2 - 2x - 3 & \text{if } x \geq 3. \end{cases}$$



Example 2.2.3: Solve $|x-3| = 2x$.

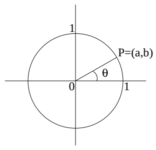


From the graph we see that the solution occurs when $x < 3$. Therefore we need

$$3 - x = 2x$$

with $x < 3$, i.e. $x = 1$.

2.3 Trigonometric functions



We define

$$\sin\theta = b \qquad \cos\theta = a$$

for $\theta \in \mathbb{R}$, and

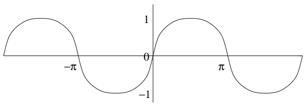
$$\tan\theta = \frac{b}{a}$$

for $\theta \in \mathbb{R}$ with $\theta \neq \left(n + \frac{1}{2}\right)\pi$ for some $n \in \mathbb{Z}$.

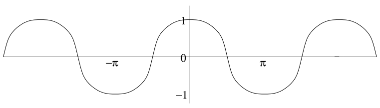
Note: (i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
(ii) We use radians for angles. 2π radians equals 360 degrees.
(iii) Positive angles are measured anticlockwise.

The graphs of these functions are:

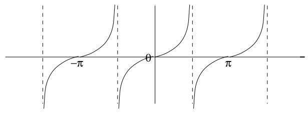
$y = \sin\theta$



$y = \cos\theta$



$y = \tan\theta$



We define

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

wherever these functions are defined, and set $\cot\frac{\pi}{2} = 0$.

A function is periodic of period t if

$$f(x+t) = f(x)$$

for all $x \in D_f$ and t is the least positive number for which this occurs.

A function is even if

$$f(-x) = f(x)$$

for all $x \in D_f$ and odd if

$$f(-x) = -f(x)$$

for all $x \in D_f$.

Here is a summary of the basic properties of trigonometric functions

Function	Domain	Range	Period	Zeros	Odd/Even
sin	$\mathbb{R}$	$\lvert y\rvert \le 1$	2π	$n\pi$	O
cos	$\mathbb{R}$	$\lvert y\rvert \le 1$	2π	$\left(\frac{2n+1}{2}\right)\pi$	E
tan	$\theta \neq \left(\frac{2n+1}{2}\right)\pi$	$\mathbb{R}$	π	$n\pi$	O
cosec	$\theta \neq n\pi$	$\lvert y\rvert \ge 1$	2π	–	O
sec	$\theta \neq \left(\frac{2n+1}{2}\right)\pi$	$\lvert y\rvert \ge 1$	2π	–	E
cot	$\theta \neq n\pi$	$\mathbb{R}$	π	$\left(\frac{2n+1}{2}\right)\pi$	O

You must memorise the following values:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–

You must also know all of the following identities:

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) \qquad \cot(x) = \tan\left(\frac{\pi}{2} - x\right)$$

$$\cos^2 x + \sin^2 x = 1$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$
$$1 + \tan^2 x = \sec^2 x$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

(From these you can work out $\sin(x-y)$ etc.)

Special cases of these last equations which should also be known are:

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

You should also know:

$$\sin x + \sin y = 2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right)$$

This last pair of equations can be derived from the preceding sets. For example, let $x = p + q$ and $y = p - q$. Then

$$\sin x + \sin y = \sin(p+q) + \sin(p-q).$$

The righthand side equals

$$\sin p\cos q + \cos p \sin q - \cos p \sin q + \sin p \cos q$$

which equals

$$2\sin p\cos q = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

Example 2.3.1: Express $\sin 3\theta$ in terms of $\sin\theta$.

$$\begin{aligned}\sin 3\theta &= \sin(\theta+2\theta)\\ &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta\\ &= \sin\theta(\cos^2\theta - \sin^2\theta)\\ &\quad +2\cos\theta\sin\theta\cos\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

When solving any trigonometric equation, we ultimately reduce to solving some equation of the form

$$f(\theta) = a$$

where f is a trigonometric function such as cos, sin, or tan. Thus we must know the general solution to such equations.

As the functions are periodic of period 2π (respectively π) for cos and sin (respectively tan), it is enough to find all solutions in some 2π period (respectively π period).

For sin, if θ is a solution then so is $\pi - \theta$.
For cos if θ is a solution then so is $-\theta$.
Tan is injective on the domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so has only one solution in each period.

In summary, the general solutions (which are to be memorised) in terms of a particular solution θ are:

sin	$\theta + 2n\pi$ or $\pi - \theta + 2n\pi$	with $n \in \mathbb{Z}$
cos	$\pm\theta + 2n\pi$	with $n \in \mathbb{Z}$
tan	$\theta + n\pi$	with $n \in \mathbb{Z}$

Example 2.3.2: Find the general solution to $\cos\theta = \frac{1}{\sqrt{2}}$.

One solution is $\theta = \frac{\pi}{4}$, so general solution is

$$\theta = \pm\frac{\pi}{4} + 2n\pi \text{ with } n \in \mathbb{Z}.$$

Example 2.3.3: Find all solutions to $\sin 2\theta = -\frac{\sqrt{3}}{2}$ with $-\pi \le \theta \le 3\pi$.

One solution is $2\theta = -\frac{\pi}{3}$, and so the general solution is

$$2\theta = -\frac{\pi}{3} + 2n\pi \quad \text{or} \quad 2\theta = \frac{4\pi}{3} + 2n\pi \qquad \text{with } n \in \mathbb{Z}.$$

Therefore

$$\theta = -\frac{\pi}{6} + n\pi \quad \text{or} \quad \theta = \frac{4\pi}{6} + n\pi \qquad \text{with } n \in \mathbb{Z}.$$

In the required range θ takes the values

$$-\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}.$$

Example 2.3.4: Solve $2\cos^2 2\theta - \sin 2\theta = 1$ for $0 \le \theta \le 2\pi$.

$$2\cos^2 2\theta - \sin 2\theta - 1 = 2(1-\sin^2 2\theta) - \sin 2\theta - 1$$

and so we require

$$(2\sin 2\theta - 1)(\sin 2\theta + 1) = 0.$$

This has solutions $\sin 2\theta = \frac{1}{2}$ and -1. Want $0 \le 2\theta \le 4\pi$. For $\sin 2\theta = \frac{1}{2}$ have

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

and for $\sin 2\theta = -1$ have

$$2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}.$$

Therefore

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{3\pi}{4}, \frac{7\pi}{4}.$$

A function of the form $a\cos\theta + b\sin\theta$ can be rewritten in either of the forms $R\cos(\theta - \alpha)$ or $R\sin(\theta + \alpha)$ for suitable choices of $R \ge 0$ and $-\frac{\pi}{2} \le \alpha < \frac{\pi}{2}$. Suppose

$$\begin{aligned}a\cos\theta + b\sin\theta &= R\cos(\theta - \alpha)\\ &= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha.\end{aligned}$$

Comparing coefficients we have

$$a = R\cos\alpha \quad \text{and} \quad b = R\sin\alpha.$$

Therefore

$$R^2(\cos^2\alpha + \sin^2\alpha) = R^2 = a^2 + b^2$$

and so $R = \sqrt{a^2 + b^2}$. Then

$$\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{b}{a}$$

and so $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

Similarly

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\sin\left(\theta + \tan^{-1}\left(\frac{a}{b}\right)\right).$$

Example 2.3.5: Find the general solution of the equation

$$\sqrt{3}\cos x + \sin x = 1.$$

Let $\sqrt{3}\cos x + \sin x = R\cos(x-\alpha)$ with $R > 0$ and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. By the above we have

$$R = \sqrt{1+3} \quad \text{and} \quad \tan\alpha = \frac{1}{\sqrt{3}}$$

which implies that $R = 2$ and $\alpha = \frac{\pi}{6}$. Thus we have to solve

$$2\cos\left(x - \frac{\pi}{6}\right) = 1.$$

This has general solution

$$x - \frac{\pi}{6} = \pm\frac{\pi}{3} + 2n\pi \qquad \text{with } n \in \mathbb{Z}.$$

There is a simple method for solving an equation of the form

$$\cos a\theta = \cos b\theta.$$

By the general form of the solution to cos we must have

$$a\theta = 2n\pi \pm b\theta$$

and so

$$\theta = \frac{2n\pi}{a \pm b} \qquad \text{with } n \in \mathbb{Z}.$$

Similar results hold for

$$\sin a\theta = \sin b\theta$$

and

$$\tan a\theta = \tan b\theta.$$

This method works when both sides of the equation involve the same function. Sometimes we will have to first rearrange to ensure this.

Example 2.3.6: Find the general solution of $\cos 2\theta = \sin\theta$.

$\sin\theta = \cos(\frac{\pi}{2} - \theta)$ and so $\cos(2\theta) = \cos(\frac{\pi}{2} - \theta)$. Therefore

$$2\theta = 2n\pi \pm \left(\frac{\pi}{2} - \theta\right) \qquad \text{with } n \in \mathbb{Z}.$$

Rearranging, we find that

$$\theta = \frac{2n\pi}{3} + \frac{\pi}{6} \qquad \text{or} \qquad \theta = 2n\pi - \frac{\pi}{2} \qquad \text{with } n \in \mathbb{Z}.$$