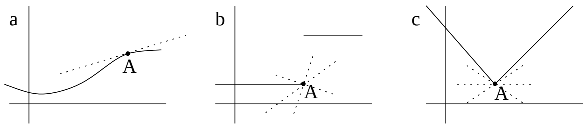
4. Calculus I: Differentiation

4.1 The derivative of a function

Suppose we are given a curve with a point A lying on it. If the curve is 'smooth' at A then we can find a unique tangent to the curve at A:

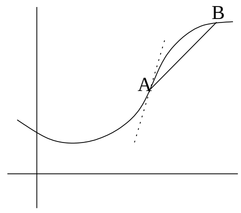


Here the curve in (a) is smooth at A, but the curves in (b) and (c) are not.

If the tangent is unique then the gradient of the curve at A is defined to be the gradient of the tangent to the curve at A.

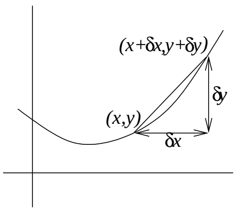
The process of finding the general gradient function for a curve is called differentiation.

Consider the chord AB. As B gets closer to A, the gradient of the chord gets closer to the gradient of the tangent at A.



For $y = f(x)$, the gradient function is defined by

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right).$$



We denote the gradient function by $\frac{dy}{dx}$ or $f'(x)$, and call it the derivative of f. This is not the formal definition of the derivative, as we have not explained exactly what we mean by the limit as $\delta x \to 0$. But this intuitive definition will be sufficient for the basic functions which we consider.

Example 4.1.1: Take $f(x) = c$, a constant function.

At every x the gradient is 0, so $f'(x) = 0$ for all x.
Or

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{c - c}{\delta x} = 0.$$

Example 4.1.2: Take $f(x) = ax$.

At every x the gradient is a, so $f'(x) = a$ for all x.
Or

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{a(x+\delta x) - ax}{\delta x} = \frac{a\delta x}{\delta x} = a.$$

Example 4.1.3: Take $f(x) = x^2$.

Now we need to consider the second formulation, as we cannot simply read the gradient off from the graph.

$$\begin{aligned} \frac{f(x+\delta x) - f(x)}{\delta x} &= \frac{(x+\delta x)^2 - x^2}{\delta x} \\ &= \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x} \\ &= \frac{\delta x(2x + \delta x)}{\delta x} = 2x + \delta x. \end{aligned}$$

The limit as δx tends to 0 is $2x$, so $f'(x) = 2x$.

Example 4.1.4: Take $f(x) = \frac{1}{x}$.

$$\begin{aligned} \frac{f(x+\delta x) - f(x)}{\delta x} &= \frac{1}{\delta x} \left(\frac{1}{x+\delta x} - \frac{1}{x} \right) \\ &= \frac{x - (x+\delta x)}{(\delta x)(x+\delta x)x} \\ &= \frac{-\delta x}{(\delta x)(x+\delta x)x} = \frac{-1}{(x+\delta x)x}. \end{aligned}$$

The limit as δx tends to 0 is $-\frac{1}{x^2}$, so $f'(x) = -\frac{1}{x^2}$.

Example 4.1.5: Take $f(x) = x^n$ with $n \in \mathbb{N}$ and $n > 1$.

Recall that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$$

and so

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}$$

where the sum has n terms. As $a \to b$ we have

$$\lim_{a \to b} \left(\frac{a^n - b^n}{a - b} \right) = \lim_{a \to b} (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}) = nb^{n-1}.$$

If $a = x + \delta x$ and $b = x$ then

$$\lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right) = \lim_{a \to b} \left(\frac{a^n - b^n}{a - b} \right) = nb^{n-1} = nx^{n-1}.$$

Hence $f'(x) = nx^{n-1}$.

Example 4.1.6: $f(x) = \sin x$.

We use the identity for $\sin A + \sin B$.

$$f(x+\delta x) - f(x) = 2\sin\left(\frac{\delta x}{2}\right)\cos\left(x + \frac{\delta x}{2}\right)$$

and so

$$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}} \cos\left(x + \frac{\delta x}{2}\right).$$

We need the following fact (which we will not prove here):

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

and so

$$f'(x) = \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}} \cos\left(x + \frac{\delta x}{2}\right) = \cos(x).$$

Some standard derivatives, which must be memorised:

$f(x)$	$f'(x)$
x^k	kx^{k-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

Some of these results can be derived from the results in the following sections, or from first principles. However it is much more efficient to know them.

4.2 Differentiation of compound functions

Once we know a few basic derivatives, we can determine many others using the following rules:
Let $u(x)$ and $v(x)$ be functions of x, and a and b be constants.

	Function	Derivative	
Sum and difference	$au \pm bv$	$a\frac{du}{dx} \pm b\frac{dv}{dx}$	
Product	uv	$v\frac{du}{dx} + u\frac{dv}{dx}$	
Quotient	$\frac{u}{v}$	$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	
Composite	$u(v(x))$	$\frac{du}{dz} \cdot \frac{dz}{dx}$	where $z = v(x)$.

The final rule above is known as the chain rule and has the following special case

$$u(ax+b) \qquad a\frac{du}{dx}(ax+b)$$

For example, the derivative of $\sin(ax+b)$ is $a\cos(ax+b)$.

Example 4.2.1: Differentiate

$$y = 2x^5 - 3x^3 + \frac{4}{x^2}.$$

$$\frac{dy}{dx} = 10x^4 - 9x^2 - \frac{8}{x^3}.$$

Example 4.2.2: Differentiate

$$y = \frac{x^2-1}{x^2+1}.$$

$$\frac{dy}{dx} = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Example 4.2.3: Differentiate

$$y = x^2 \ln(x+3).$$

$$\frac{dy}{dx} = 2x\ln(x+3) + \frac{x^2}{x+3}.$$

Example 4.2.4: Differentiate $y = e^{5x}$.

Set $z = 5x$, then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^z 5 = 5e^{5x}.$$

Example 4.2.5: Differentiate $y = 4\sin(2x+3)$.

Set $z = 2x+3$, then

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 4\cos(z)2 = 8\cos(2x+3).$$

As we have already noted, some of the standard derivatives can be deduced from the others.

Example 4.2.6: Differentiate

$$y = \tan x = \frac{\sin x}{\cos x}.$$

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Example 4.2.7: $y = \operatorname{cosec} x = \frac{1}{\sin x}$.

$$\frac{dy}{dx} = \frac{\sin x.(0) - 1.\cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$$

Example 4.2.8: $y = \ln(x + \sqrt{x^2+1})$, i.e. $y = \ln u$ where $u = x + \sqrt{x^2+1}$.

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} \quad \text{and} \quad \frac{du}{dx} = 1 + \frac{(x^2+1)^{-\frac{1}{2}}}{2}.2x$$

so

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$
$$= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}.$$

Example 4.2.9: $y = x^x$.

We have $y = (e^{\ln x})^x = e^{(x\ln x)}$, i.e. $y = e^u$ where $u = x\ln x$.

$$\frac{dy}{dx} = e^u\frac{du}{dx} = e^{x\ln x}(\ln(x)+1) = x^x(\ln(x)+1).$$

4.3 Higher derivatives

The derivative $\frac{dy}{dx}$ is itself a function, so we can consider its derivative. If $y = f(x)$ then we denote the second derivative, i.e. the derivative of $\frac{dy}{dx}$ with respect to x, by $\frac{d^2y}{dx^2}$ or $f''(x)$. We can also calculate the higher derivatives $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

Example 4.3.1: $y = \ln(1+x^2)$.

Let $z = \frac{dy}{dx} = \frac{2x}{1+x^2}$.

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = \frac{(1+x^2).2 - 2x(2x)}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}.$$

Example 4.3.2: Show that $y = e^{-x}\sin(2x)$ satisfies

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

$$\frac{dy}{dx} = -e^{-x}\sin 2x + 2e^{-x}\cos 2x = e^{-x}(2\cos 2x - \sin 2x)$$

$$\frac{d^2y}{dx^2} = -e^{-x}(2\cos 2x - \sin 2x) + e^{-x}(-4\sin 2x - 2\cos 2x)$$
$$= e^{-x}(-3\sin 2x - 4\cos 2x).$$

Writing s for $\sin 2x$ and c for $\cos 2x$ we have

$$y'' + 2y' + 5y = e^{-x}(-3s - 4c - 2s + 4c + 5s) = 0.$$

Example 4.3.3: Evaluate

$$\frac{d^3}{dx^3}\left(\frac{1+3x^2}{(1+x)^2(1+3x)}\right)$$

at $x = 0$.

We could use the quotient rule, but this will get complicated. Instead we use partial fractions.

$$y = \frac{1+3x^2}{(1+x)^2(1+3x)} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1+3x}.$$

We obtain (check!) $A = 0$, $B = -2$, and $C = 3$.

Now

$$\frac{dy}{dx} = \frac{4}{(1+x)^3} - \frac{9}{(1+3x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-12}{(1+x)^4} + \frac{54}{(1+3x)^3}$$

$$\frac{d^3y}{dx^3} = \frac{48}{(1+x)^5} - \frac{54\times 9}{(1+3x)^4}$$

and substituting $x = 0$ we obtain that

$$\frac{d^3y}{dx^3}(0) = 48 - 486 = -438.$$

Generally it is hard to give a simple formula for the nth derivative of a function. However, in some cases it is possible. The following can be proved by induction.

Example 4.3.4: $y = e^{ax}$.

$$\frac{dy}{dx} = ae^{ax} \quad \text{and} \quad \frac{d^2y}{dx^2} = a^2e^{ax}.$$

We can show that

$$\frac{d^ny}{dx^n} = a^ne^{ax}.$$

Example 4.3.5: $y = \sin(ax)$.

$$\begin{aligned} y' &= a\cos(ax) &&= a\sin(ax + \tfrac{\pi}{2}) \\ y'' &= -a^2\sin(ax) &&= a^2\sin(ax + \pi) \\ y''' &= -a^3\cos(ax) &&= a^3\sin(ax + \tfrac{3\pi}{2}) \\ y^{(iv)} &= a^4\sin(ax) &&= a^4\sin(ax + 2\pi). \end{aligned}$$

We can show that

$$\frac{d^ny}{dx^n} = a^n\sin(ax + \frac{n\pi}{2}).$$

Sometimes it is useful to use the Leibnitz rule, which gives a formula for the nth derivative of a product of functions.

$$\frac{d^n}{dx^n}(fg) = \frac{d^n}{dx^n}(f)g + \binom{n}{1}\frac{d^{n-1}}{dx^{n-1}}(f)\frac{d}{dx}(g) + \binom{n}{2}\frac{d^{n-2}}{dx^{n-2}}(f)\frac{d^2}{dx^2}(g) + \cdots$$
$$\cdots + \binom{n}{n-1}\frac{d}{dx}(f)\frac{d^{n-1}}{dx^{n-1}}(g) + f\frac{d^n}{dx^n}(g).$$

So for example

$$\frac{d^3}{dx^3}(fg) = \frac{d^3}{dx^3}(f)g + 3\frac{d^2}{dx^2}(f)\frac{d}{dx}(g) + 3\frac{d}{dx}(f)\frac{d^2}{dx^2}(g) + f\frac{d^3}{dx^3}(g).$$

As can be seen, this is very similar to the binomial theorem, and can also be proved by induction (using the product rule).

4.4 Differentiating implicit functions

Sometimes we cannot rearrange a function into the form $y = f(x)$, or we may wish to consider the original form anyway (for example, because it is simpler). However, we may still wish to differentiate with respect to x.

Given a function $g(y)$ we have from the chain rule

$$\frac{d}{dx}(g(y)) = \frac{d}{dy}(g(y))\frac{dy}{dx}.$$

Example 4.4.1: $x^2 + 3xy^2 - y^4 = 2$.

$$\frac{d}{dx}(x^2 + 3xy^2 - y^4) = \frac{d}{dx}(2) = 0.$$

Therefore we have

$$\begin{aligned} 2x + \frac{d}{dx}(3xy^2) - \frac{d}{dx}(y^4) &= 0 \\ 2x + 3y^2 + 3x\frac{d}{dx}(y^2) - 4y^3\frac{dy}{dx} &= 0 \\ 2x + 3y^2 + 6xy\frac{dy}{dx} - 4y^3\frac{dy}{dx} &= 0. \end{aligned}$$

Example 4.4.2: $\frac{2}{x^2} + \frac{3}{y^2} = \frac{1}{2}$.

$$\frac{d}{dx}(\frac{2}{x^2} + \frac{3}{y^2}) = \frac{d}{dx}(\frac{1}{2}) = 0.$$

Therefore we have

$$\begin{aligned} -\frac{4}{x^3} + \frac{d}{dx}\left(\frac{3}{y^2}\right) &= 0 \\ -\frac{4}{x^3} - \frac{6}{y^3}\frac{dy}{dx} &= 0. \end{aligned}$$

4.5 Differentiating parametric equations

Sometimes there is no easy way to express the relationship between x and y directly in a single equation. In such cases it may be possible to express the relationship between them by writing each in terms of a third variable. We call such equations parametric equations as both x and y depend on a common parameter.

Example 4.5.1: $x = t^3 \qquad y = t^2 - 4t + 2.$

Although we can write this in the form

$$y = x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 2$$

the parametric version is easier to work with.

To differentiate a parametric equation in the variable t we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Example 4.5.1: (Continued.)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t - 4 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-4}{3t^2}.$$

Example 4.5.2: Find the second derivative with respect to x of

$$x = \sin\theta \qquad y = \cos 2\theta.$$

We have

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = -2\sin 2\theta.$$

Therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2\theta}{\cos\theta} = -4\sin\theta.$$

Now

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(-4\sin\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(-4\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{-4\cos\theta}{\cos\theta} = -4.$$

Note: The rules so far may suggest that derivatives can be treated just like fractions. However

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\frac{\mathrm{d}^2t}{\mathrm{d}x^2}$$

in general. Moreover

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \neq \left(\frac{\mathrm{d}^2x}{\mathrm{d}y^2}\right)^{-1}.$$

4.6 Tangents and normals to curves

We have already defined the value of the derivative f' of a function f at a point x_0 to be the gradient of f at x_0. Thus we can easily use the derivative to write down the equation of the tangent to that point. Using the equation for a line passing through $(x_0, f(x_0))$ we have that the tangent to f at x_0 is

$$y - f(x_0) = \frac{\mathrm{d}y}{\mathrm{d}x}(x_0)(x - x_0).$$

The normal to f at x_0 is the line passing through $(x_0, f(x_0))$ perpendicular to the tangent. This has equation

$$y - f(x_0) = \frac{-1}{\frac{\mathrm{d}y}{\mathrm{d}x}(x_0)}(x - x_0)$$

(when this makes sense).

Example 4.6.1: Find the equation of the tangent and normal to the curve

$$y = x^2 - 6x + 5$$

at the point $(2, -3)$.

We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6$$

and hence $\frac{\mathrm{d}y}{\mathrm{d}x}(2) = 4 - 6 = -2$. Hence the equation of the tangent is

$$y + 3 = -2(x - 2) \quad \text{i.e.} \quad y = -2x + 1.$$

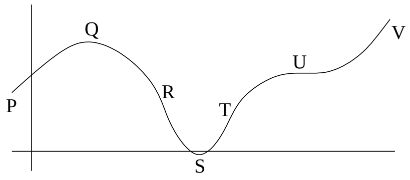
The gradient of the normal is $\frac{-1}{-2} = \frac{1}{2}$, and hence the equation of the normal is

$$y + 3 = \frac{1}{2}(x - 2) \quad \text{i.e.} \quad y = \frac{x}{2} - 4.$$

4.7 Stationary points and points of inflexion

We can tell a lot about a function from its derivatives.

Example 4.7.1:



If $f'(x) > 0$ for $a < x < b$ then f is increasing on $a < x < b$

e.g. arcs PQ, SU, UV.

If $f'(x) < 0$ for $a < x < b$ then f is decreasing on $a < x < b$

e.g. arc QS.

A stationary point on a curve $y = f(x)$ is a point $(x_0, f(x_0))$ such that $f'(x_0) = 0$. These come in various forms:

Type	Test $f'(x)$	Test $f''(x)$
Local maximum	Changes from $+$ to $-$	$-$ve
Local minimum	Changes from $-$ to $+$	$+$ve
Point of inflexion	No sign change	(see below)

e.g. Q is a max, S is a min, U is a point of inflexion.

A point of inflexion is one where $f''(x_0) = 0$ and f'' changes sign at x_0.

e.g. R, T, U.

If $f''(x) > 0$ for $a < x < b$ then f is concave up on $a < x < b$

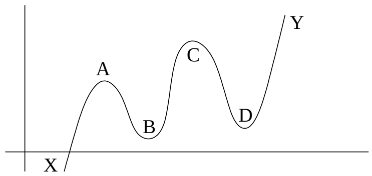
e.g. arc RST.

If $f''(x) < 0$ for $a < x < b$ then f is concave down on $a < x < b$

e.g. arc PQR.

Note that the maxima and minima above are only local. This means that in a small region about the given point they are extremal values, but perhaps not over the whole curve. Extremal values for the whole curve are called global maxima or minima.

Example 4.7.2: Consider the function f on the domain $X \leq x \leq Y$ given by the graph



Both A and C are local maxima, and B and D are local minima. However the global maximum is at Y and the global minimum at X.

Example 4.7.3: Find the stationary values and points of inflexion of

$$y = 3x^4 + 8x^3 - 6x^2 - 24x + 2.$$

We have

$$\frac{dy}{dx} = 12x^3 + 24x^2 - 12x - 24$$

and

$$\frac{d^2y}{dx^2} = 36x^2 + 48x - 12.$$

Stationary points when $\frac{dy}{dx} = 0$, i.e. (check) $x = 1, -1, -2$.

	y'	y''	
$(1, -17)$	$-0+$	72	Min
$(-1, 15)$	$+0-$	-24	Max
$(-2, 10)$	$-0+$	36	Min

Points of inflexion at $x = \frac{1}{3}(-2 \pm \sqrt{7})$, i.e. $(x, y) \approx (0.22, -3.36)$ and $(x, y) \approx (-1.55, 12.32)$.