Example 4.7.4: Find the stationary points of the curve

$$f(x) = 6\ln\left(\frac{x}{7}\right) + (x-1)(x-7).$$

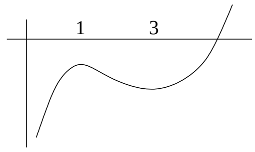
Deduce that $f(x) = 0$ has only one solution, and state its value.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{x} + 2x - 8 \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{6}{x^2} + 2.$$

We have $f'(x) = 0$ when $2x^2 - 8x + 6 = 0$, i.e. $x = 1$ or 3.

$f''(1) = -4$ so there is a local max at $(1, -6\ln 7)$.
$f''(3) = \frac{4}{3}$ so there is a local min at $(3, -6\ln(\frac{7}{3}) - 8)$.

For large x the function f is large and positive. Therefore the curve is of the form



It cannot cross the x-axis again as there are no other turning points, so $f(x) = 0$ has only one solution. By inspection, $x = 7$ is a root.

Example 4.7.5: Find the least value of

$$y = a^2\sec^2 x + b^2\operatorname{cosec}^2 x$$

where a and b are positive constants and $0 < x < \frac{\pi}{2}$.

$$\begin{aligned}\tfrac{\mathrm{d}y}{\mathrm{d}x} &= 2a^2\sec x(\sec x\tan x) + 2b^2\operatorname{cosec} x(-\operatorname{cosec} x\cot x)\\ &= 2a^2\sec^2 x\tan x - 2b^2\operatorname{cosec}^2 x\cot x\\ &= 2a^2\frac{\sin x}{\cos^3 x} - 2b^2\frac{\cos x}{\sin^3 x}\\ &= \frac{2a^2\sin^4 x - 2b^2\cos^4 x}{\cos^3 x\sin^3 x}.\end{aligned}$$

Stationary point are where $y' = 0$, i.e. where

$$2a^2\sin^4 x - 2b^2\cos^4 x = 0.$$

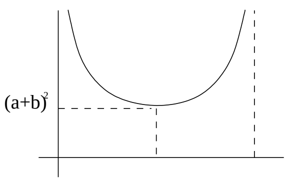
This can be rearranged to give

$$\tan^4 x = \frac{b^2}{a^2} \quad \text{or} \quad \tan^2 x = \frac{b}{a}.$$

Since $0 < x < \frac{\pi}{2}$ we have $\tan x > 0$, and so $\tan x = \sqrt{b/a}$, and there is precisely one stationary point.

Since $y \to \infty$ as $x \to 0$ or $x \to \frac{\pi}{2}$, the stationary point must be a minimum. Substituting for $\tan x$ in y gives

$$\begin{aligned}y &= a^2(1+\tan^2 x) + b^2(1+\cot^2 x)\\ &= a^2\left(1+\frac{b}{a}\right) + b^2\left(1+\frac{a}{b}\right)\\ &= a^2 + 2ab + b^2 = (a+b)^2\end{aligned}$$



5. Calculus II: Integration

5.1 Basic theory

We will define the integral of a function $f(x)$ to be its antiderivative:

$$\int f(x)\,dx = F(x) + C$$

where C is a constant and $F(x)$ is a function with $\frac{dF}{dx} = f(x)$. Any two functions F and G with $\frac{dF}{dx} = \frac{dG}{dx} = f(x)$ must satisfy $\frac{d}{dx}(F-G) = 0$, i.e. $F - G$ is some constant function. Thus the integral is only defined up to the undetermined constant C.

From our standard results for differentiation we deduce the following integrals, which must be memorised.

$f(x)$	$\int f(x)\,dx$
x^k $(k \neq -1)$	$\frac{1}{k+1}x^{k+1} + C$
x^{-1}	$\ln x + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$	$-\ln(\cos x) + C$

There are obvious extensions of these results, replacing x by $ax + b$. For example, for $k \neq -1$ we have

$$\int (ax+b)^k\,dx = \frac{(ax+b)^{k+1}}{a(k+1)} + C$$

and

$$\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + C.$$

etc. We also have for functions f and g and constants a and b that

$$\int af + bg\,dx = a\int f\,dx + b\int g\,dx.$$

Example 5.1.1:

$$\int x^7 + \frac{3}{x^2} - \sqrt{x}dx = \int x^7 dx + 3\int x^{-2}dx - \int x^{\frac{1}{2}} dx$$
$$= \frac{x^8}{8} - \frac{3}{x} - \frac{2}{3}x^{\frac{3}{2}} + C.$$

Example 5.1.2:

$$\int \frac{1}{(2x+3)^4}dx = \frac{(2x+3)^{-3}}{(-3).2} + C = \frac{-1}{6(2x+3)^3} + C.$$

For more complicated rational functions we usually simplify first using partial fractions.

Example 5.1.3:

$$\int \frac{1}{(x-1)(x-2)}dx = \int \frac{-1}{(x-1)} + \frac{1}{x-2}dx$$
$$= -\ln(x-1) + \ln(x-2) + C = \ln\left(\frac{x-2}{x-1}\right) + C.$$

Example 5.1.4:

$$\int \frac{1+3x^2}{(1+x)^2(1+3x)}dx = \int \frac{-2}{(1+x)^2} + \frac{3}{1+3x}dx = \frac{2}{1+x} + \ln(1+3x) + C.$$

Example 5.1.5:
$$\int \sin 5x \, dx = -\frac{1}{5}\cos 5x + C.$$

For more complicated integrals involving trigonometric functions, we typically use standard identities to simplify the integral.

Example 5.1.6:

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C.$$

Example 5.1.7:

$$\int \sin 3x \cos x dx = \int \frac{\sin(3x+x) + \sin(3x-x)}{2}dx$$
$$= \int \frac{1}{2}(\sin 4x + \sin 2x)dx = -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C.$$

Sometimes it is not so easy to spot the integral of a function.

Example 5.1.8: $\int 2xe^{x^2} dx$.

This does not correspond to one of our standard integrals. However, by inspection we can observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{x^2}) = 2xe^{x^2}$$

using the chain rule, and hence

$$\int 2xe^{x^2} dx = e^{x^2} + C.$$

We would like to formalise this procedure.

5.2 Method of substitution

Recall the chain rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))g'(x).$$

Integrating both sides we obtain

$$\int f'(g(x))g'(x)\, dx = f(g(x)) + C.$$

Writing $u = g(x)$ this becomes

$$\int f'(u)\frac{\mathrm{d}u}{\mathrm{d}x}\, dx = f(u) + C$$

and so we have

$$\int f'(g(x))g'(x)\, dx = \int f'(u)\, du$$

where $u = g(x)$.

Example 5.2.1: We return to example 5.1.8, and recalculate

$$\int 2xe^{x^2}\, dx.$$

Let $u = x^2$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$. Then

$$\int 2xe^{x^2}\, dx = \int e^u \frac{\mathrm{d}u}{\mathrm{d}x}\, dx = \int e^u\, du = e^u + C = e^{x^2} + C.$$

Example 5.2.2: Integrate

$$\int x^2(x^3+1)^{\frac{3}{2}}\,dx.$$

Let $u = x^3+1$, so $\frac{du}{dx} = 3x^2$. Then

$$\int x^2(x^3+1)^{\frac{3}{2}}\,dx = \int \frac{u^{\frac{3}{2}}}{3}\frac{\mathrm{d}u}{\mathrm{d}x}\,dx$$
$$= \int \frac{u^{\frac{3}{2}}}{3}\,du = \frac{2}{15}u^{\frac{5}{2}} + C = \frac{2}{15}(x^3+1)^{\frac{5}{2}} + C.$$

Example 5.2.3: Integrate

$$\int \sin^4 x\cos x\,dx.$$

Let $u = \sin x$, so $\frac{du}{dx} = \cos x$. Then

$$\int \sin^4 x\cos x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$$

Example 5.2.4: Integrate

$$\int \tan x\,dx.$$

First note that

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

Let $u = \cos x$, so $\frac{du}{dx} = -\sin x$. Then

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln(u)+C = -\ln(\cos x)+C = \ln(\sec x)+C.$$

5.3 Inverse substitution

In the last section we substituted

$$f'(g(x)) \longrightarrow f'(u)$$
$$g'(x)\,dx \longrightarrow du.$$

Next we consider the inverse substitution. Replacing f' by h and interchanging the roles of x and u we have

$$\int h(g(u))g'(u)\,du = \int h(x)\,dx$$

where $x = g(u)$. Therefore we can substitute

$$h(x) \longrightarrow h(g(u))$$
$$dx \longrightarrow g'(u)\,du = \frac{\mathrm{d}x}{\mathrm{d}u}\,du.$$

Example 5.3.1: Integrate

$$\int \frac{1}{1+\sqrt{x}}\,dx.$$

Let $\sqrt{x} = u$, so $x = u^2$ and $\frac{dx}{du} = 2u$. Then

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+u}2u\,du$$
$$= \int 2 - \frac{2}{1+u}\,du$$
$$= 2u - 2\ln(1+u) + C = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$$

Example 5.3.2: Integrate

$$\int \frac{x-2}{\sqrt{2x+3}}\,dx.$$

Let $u = \sqrt{2x+3}$, so $2x+3 = u^2$ and $\frac{dx}{du} = u$. Then

$$\int \frac{x-2}{\sqrt{2x+3}}\,dx = \int \frac{\frac{1}{2}(u^2-3)-2}{u}u\,du$$
$$= \int \frac{1}{2}(u^2-7)\,du$$
$$= \frac{u^3}{6} - \frac{7u}{2} + C = \frac{u}{6}(u^2-21) + C$$
$$= \frac{\sqrt{2x+3}}{6}(2x-18) + C.$$

Example 5.3.3: Integrate

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx.$$

Let $x = 2\sin\theta$, so $\frac{dx}{d\theta} = 2\cos\theta$, and $4-x^2 = 4\cos^2\theta$. Then

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx = \int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta$$
$$= \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$$
$$= \frac{1}{4}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C.$$

5.4 Integration by parts

Recall the rule for differentiating a product of functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}.v + u.\frac{\mathrm{d}v}{\mathrm{d}x}.$$

Using the antiderivative this becomes

$$uv = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,dx + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,dx.$$

Therefore

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,dx = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,dx.$$

Example 5.4.1: Calculate

$$\int x\cos x\,dx.$$

Let $u = x$ and $\frac{dv}{dx} = \cos x$. Then $\frac{du}{dx} = 1$ and $v = \sin x$.

$$\int x\cos x\,dx = x\sin x - \textstyle\int(\sin x).1\,dx$$
$$= x\sin x + \cos x + C.$$

Example 5.4.2: Calculate

$$S = \int x^2 e^{3x}\,dx.$$

Let $u = x^2$ and $\frac{dv}{dx} = e^{3x}$. Then $\frac{du}{dx} = 2x$ and $v = \frac{1}{3}e^{3x}$.

$$S = \frac{x^2}{3}e^{3x} - \int \frac{2x}{3}e^{3x}\,dx = \frac{x^2}{3}e^{3x} - T.$$

Now use integration by parts again to determine T

Let $u = \frac{2x}{3}$ and $\frac{dv}{dx} = e^{3x}$. Then $\frac{du}{dx} = \frac{2}{3}$ and $v = \frac{1}{3}e^{3x}$.

$$\begin{aligned} T &= \frac{2x}{3}\frac{e^{3x}}{3} - \int \frac{2}{9}e^{3x}\,dx \\ &= \frac{2x}{9}e^{3x} - \frac{2}{27}e^{3x} + C. \end{aligned}$$

So

$$S = \left(\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right)e^{3x} + C.$$

Using this method we can integrate another of our standard functions.

Example 5.4.3: Calculate

$$\int \ln(x)\,dx.$$

Let $u = \ln(x)$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

$$\begin{aligned} \int \ln(x)\,dx &= x\ln(x) - \int \tfrac{x}{x}\,dx \\ &= x\ln(x) - x + C \end{aligned}$$

Next time we will see how integration by parts can be used in more complicated examples.

We saw in Example 5.4.2 that we sometimes need to apply integration by parts several times in the course of a single calculation.

Example 5.4.4: For $n \geq 0$ let

$$S_n = \int x^n \cos 2x\,dx.$$

Find an expression for S_n in terms of S_{n-2}, and hence evaluate S_4.

Let $u = x^n$ and $\frac{dv}{dx} = \cos 2x$. Then $\frac{du}{dx} = nx^{n-1}$ and $v = \frac{1}{2}\sin(2x)$.

Integrating by parts we have

$$\begin{aligned} \int x^n \cos 2x\,dx &= \frac{x^n}{2}\sin(2x) - \int \frac{n}{2}x^{n-1}\sin 2x\,dx \\ &= \frac{x^n}{2}\sin(2x) + \frac{n}{4}x^{n-1}\cos 2x \\ &\quad - \int \frac{n(n-1)}{4}x^{n-2}\cos 2x\,dx \\ &= \frac{x^n}{2}\sin(2x) + \frac{n}{4}x^{n-1}\cos 2x - \frac{n(n-1)}{4}S_{n-2} \end{aligned}$$

where the second equality follows using integration by parts with $u = \frac{n}{2}x^{n-1}$ and $\frac{dv}{dx} = \sin 2x$. Thus we have found a formula for S_n in terms of S_{n-2}.

Clearly $S_0 = \int \cos 2x\,dx = \frac{1}{2}\sin 2x$. Hence

$$S_2 = \frac{x^2}{2}\sin(2x) + \frac{2}{4}x\cos 2x - \frac{1}{4}\sin 2x$$

and

$$\begin{aligned} S_4 &= \frac{x^4}{2}\sin(2x) + \frac{4}{4}x^3\cos 2x \\ &\quad -3\left(\frac{x^2}{2}\sin(2x) + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x\right) \\ &= \frac{1}{4}(2x^4 - 6x^2 + 3)\sin 2x + \frac{1}{2}(2x^3 - 3x)\cos 2x. \end{aligned}$$

In some examples integration by parts does not lead to a simpler integral. However, even in these cases we can sometimes use this method to solve the original problem.

Example 5.4.5: Calculate

$$\int e^x \cos x\,dx.$$

Let $u = e^x$ and $\frac{dv}{dx} = \cos x$. Integrating by parts we obtain

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Integrating by parts again we have

$$\int e^x \cos x\,dx = e^x \sin x - \left[-e^x\cos x + \int e^x \cos x\,dx\right].$$

The final integral is identical to that we first wished to calculate, however we can now rearrange this formula to obtain

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

from which we deduce that

$$\int e^x \cos x\,dx = \frac{1}{2}(e^x \sin x + e^x \cos x)$$

5.5 The definite integral

If

$$\int g(x)\,dx = G(x) + C$$

then we define

$$\int_a^b g(x)\,dx = G(b) - G(a)$$

which we also denote by

$$\Big[G(x)\Big]_a^b.$$

Example 5.5.1:

$$\begin{aligned}\int_1^4 \frac{1}{(x+3)^2}dx &= \left[\frac{-1}{x+3}\right]_1^4 \\ &= -\frac{1}{7} - \left(-\frac{1}{4}\right) = \frac{3}{28}.\end{aligned}$$

In the next example we will apply Example 5.2.3.

Example 5.5.2:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^4 x \cos x\,dx &= \left[\frac{1}{5}\sin^5 x\right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{5} - 0 = \frac{1}{5}.\end{aligned}$$

When integrating a definite integral by substitution we must be careful to convert the limits into the new variable.

Example 5.5.3: Calculate

$$\int_0^2 \sqrt{4-x^2}\,dx.$$

Let $x = 2\sin\theta$, so $\frac{dx}{d\theta} = 2\cos\theta$. We have

$$4 - x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$$

and in changing variable we have

$$\begin{aligned} x = 0 &\longrightarrow \theta = 0 \\ x = 2 &\longrightarrow \theta = \frac{\pi}{2}.\end{aligned}$$

Combining these observations we obtain

$$\begin{aligned}\int_0^2 \sqrt{4-x^2}\,dx &= \int_0^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\,d\theta \\ &= \int_0^{\frac{\pi}{2}} 4\cos^2\theta\,d\theta \\ &= \int_0^{\frac{\pi}{2}} 2(1+\cos 2\theta)\,d\theta \\ &= \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}} \\ &= 2\left[\frac{\pi}{2} + 0 - 0 - 0\right] = \pi.\end{aligned}$$