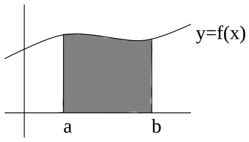
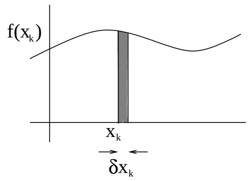
5.6 Integration as a measure of content



The area contained between the curve $y = f(x)$, the lines $x = a$ and $x = b$ (for $a < b$) and the x-axis is given by

$$\int_a^b f(x)\,dx.$$

This follows from the definition of integration as a measure:

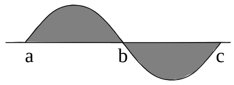


$$\text{area} \approx \sum_{k=1}^{n} f(x_k)\delta x_k$$

and the fundamental theorem of calculus which states that this definition agrees with that coming from the antiderivative.

Note that this result relies on the convention that area below the x-axis is negative. When calculating area we do not use this convention, so the answer will have to be adjusted appropriately.

So for



we have

$$\int_a^b f(x)\,dx = -\int_b^c f(x)\,dx$$

although the total area is clearly non-zero.

If $b < a$ we define

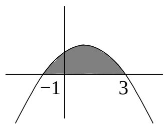
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

(This must clearly be the case from the definition of integration using the antiderivative.)

Example 5.6.1: Find the area contained between the quadratic

$$y = 3 + 2x - x^2$$

and the x-axis.



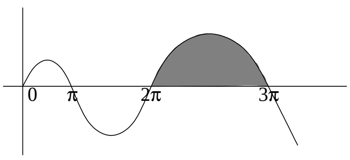
We have $y = (3 - x)(x + 1)$, and from the graph we see that

$$\begin{aligned}\text{area} &= \int_{-1}^{3} 3 + 2x - x^2\,dx\\ &= \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} = \frac{32}{3}.\end{aligned}$$

Example 5.6.2: Find the area contained in the third arc of the curve

$$y = x \sin x$$

for $x \geq 0$.



$$\begin{aligned}\text{area} &= \int_{2\pi}^{3\pi} x \sin x\,dx \qquad \text{(use integration by parts)}\\ &= \left[-x\cos x - \int(-\cos x)\,dx\right]_{2\pi}^{3\pi} = \left[-x\cos x + \sin x\right]_{2\pi}^{3\pi}\\ &= [(-3\pi)(-1) + 0] - [(-2\pi)(1) + 0] = 5\pi.\end{aligned}$$

Note that if the example had asked for the second and third arcs, we would have calculated

$$\int_{2\pi}^{3\pi} x \sin x\,dx - \int_{\pi}^{2\pi} x \sin x\,dx.$$

Example 5.6.3: Find the area enclosed by the line $y = 2x$ and the curve

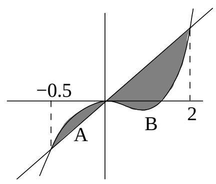
$$y = 2x^3 - 3x^2.$$

Line and curve intersect when

$$2x^3 - 3x^2 = 2x$$

i.e. when

$$x(2x + 1)(x - 2) = 0.$$



Let $y_1 = 2x$ and $y_2 = 2x^3 - 3x^2$. Then

$$\begin{aligned}\text{area A} &= \textstyle\int_{-\frac{1}{2}}^{0} y_2 - y_1\,dx = \displaystyle\int_{-\frac{1}{2}}^{0} 2x^3 - 3x^2 - 2x\,dx\\ &= \left[\frac{x^4}{2} - x^3 - x^2\right]_{-\frac{1}{2}}^{0} = \frac{3}{32}.\end{aligned}$$

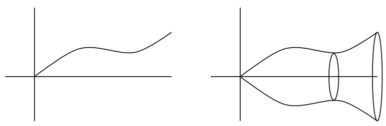
and

$$\begin{aligned}\text{area B} &= \textstyle\int_{0}^{2} y_1 - y_2\,dx = \displaystyle\int_{0}^{2} -2x^3 + 3x^2 + 2x\,dx\\ &= \left[-\frac{x^4}{2} + x^3 + x^2\right]_{0}^{2} = 4.\end{aligned}$$

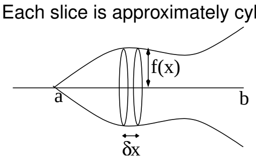
Therefore the total area is $A + B = \frac{131}{32}$.

5.7 Volume of revolution

If an area is rotated about a straight line, the 3-dimensional object formed is called a solid of revolution, and its volume is a volumne of revolution.



To calculate the volumn, divide the solid into slices perpendicular to the axis of rotation. Each slice is approximately cylindrical.



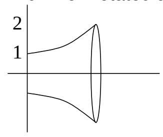
Suppose the axis of rotation is the x-axis. Each slice has radius $y = f(x)$ and thickness δx. Therefore

$$\delta V \approx \pi y^2 \delta x.$$

Now the volume of the solid is

$$V = \lim_{\delta x \to 0} \left\{ \sum_{x=a}^{x=b} \pi y^2 \delta x \right\} = \int_a^b \pi y^2 \, dx.$$

Example 5.7.1: Find the volume generated by the curve $y = x^2 + 1$ and the lines $x = 1$ and $x = 0$ when rotated about the x-axis.

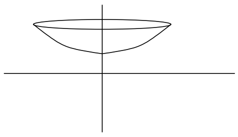


$$\delta V \approx \pi y^2 \delta x = \pi (x^2 + 1)^2 \delta x.$$

Therefore

$$V = \int_0^1 \pi (x^2+1)^2 \, dx = \int_0^1 \pi (x^4 + 2x^2 + 1) \, dx = \frac{28\pi}{15}.$$

Example 5.7.2: Find the volume generated when the area bounded by the curve $y = x^2 + 1$ (for $x \geq 0$) and the line $y = 2$ is rotated about the y-axis.



Slices are perpendicular to the y-axis, so

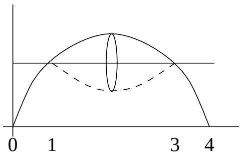
$$\delta V \approx \pi x^2 \delta y.$$

Now $x^2 = y - 1$, so

$$V = \int_1^2 \pi x^2 \, dy = \int_1^2 \pi (y-1) \, dy = \frac{\pi}{2}.$$

Example 5.7.3: The area enclosed by the curve $y = 4x - x^2$ and the line $y = 3$ is rotated about the line $y = 3$. Find the volume of the solid generated.

$y = 4x - x^2 = x(4 - x)$ and so $y = 0$ at $x = 0$ and $x = 4$. Similarly $4x - x^2 = 3$ implies that $x = 1$ or $x = 3$.



We have that

$$\delta V = \pi (y-3)^2 \delta x$$

where $y = 4x - x^2$. Therefore

$$V = \int_1^3 \pi (4x - x^2 - 3)^2 \, dx = \int_1^3 \pi (x^4 - 8x^3 + 22x^2 - 24x + 9) \, dx = \frac{16\pi}{15}.$$

There are two other integrals you may come across in the statistics course.

The mean value of $y = f(x)$ on $a \leq x \leq b$ is

$$\bar{y} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

The root mean square of $y = f(x)$ on $a \leq x \leq b$ is

$$\hat{y} = \left(\frac{1}{b-a} \int_a^b [f(x)]^2 \, dx \right)^{\frac{1}{2}}.$$

Example 5.7.4: Find the root mean square of $y = \sin x$ on $0 \leq x \leq \pi$.

$$\hat{y} = \left(\frac{1}{\pi} \int_0^\pi \sin^2 x \, dx \right)^{\frac{1}{2}} = \left(\frac{1}{\pi} \int_0^\pi \frac{1}{2}(1 - \cos 2x) \, dx \right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{\pi} \left[\frac{\pi}{2} \right] \right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

6. Calculus III: Limits

6.1 The limit of a function

We will give an informal definition of the limit of a function (a concept which we have already used in this course). For a rigorous treatment of this subject we would need a precise definition and lots of proofs, but we will make do with an informal survey of the main results and methods.

There will be a brief introduction to the formal definition of a limit at the end of the Chapter; this will be studied in greater detail in the second year Real Analysis module.

Informally, given a function f and a number c, the limit of f as x tends to c, written $\lim_{x\to c} f(x)$ is defined in the following way.

- $\lim_{x\to c} f(x) = L$
  if the values of $f(x)$ are always arbitrarily close to L provided that x is sufficiently close (but not equal) to c.
- $\lim_{x\to c} f(x) = \infty$
  if for every given real R number, $f(x)$ is always bigger than R, provided that x is sufficiently close to c.
- $\lim_{x\to c} f(x) = -\infty$
  is similar to the previous case, but replacing bigger by smaller.

Example 6.1.1:
(a)
$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$
(b)
$$\lim_{x\to \infty} \frac{1}{x} = 0.$$
(c)
$$\lim_{x\to \infty} \frac{x}{x+1} = 1.$$

Warning: Not every limit exists!

Example 6.1.2: (a)
$$\lim_{x\to\infty} \sin x$$
does not exist as the function does not tend to a single value.

(b)
$$\lim_{x\to 0}\left(\frac{1}{x}\right)$$
does not exist as $\frac{1}{x}$ is very large in modulus and positive if $x > 0$ is very small, and is a very large in modulus and negative if $x < 0$ is very small.

If the limits of f and g as x tends to c exist and are finite then we have:
$$\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x).$$
$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$$
$$\lim_{x\to c}(f(x)^n) = \left(\lim_{x\to c} f(x)\right)^n$$
for $n \in \mathbb{N}$.
$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \left(\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}\right)$$
provided that $\lim_{x\to c} g(x) \neq 0$.

Example 6.1.3:
$$\lim_{x\to 2}\left((x+1)(x+2)^4\right) = \lim_{x\to 2}(x+1)\left(\lim_{x\to 2}(x+2)\right)^2 = 3 \times 4^4 = 768.$$

Warning: We cannot use the above identities if the limits of f or g do not exist, or are $\pm\infty$.

As Example 6.1.3 should suggest, if f is continuous at $x = c$ then
$$\lim_{x\to c} f(x) = f(c).$$

However, the usefulness of limits is that they can be used to investigate the properties of functions near values where they may not be defined. Looking at the asymptotes to a given function is also related to considering limits.

6.2 Limits of quotients of functions

A common situation where limits arise is the case where we quotient one function by another, particularly if the denominator equals zero at the point of interest. We have various methods for tackling such limits (provided that they exist).

If $f(c)$ and $g(c)$ exist and $g(c) \neq 0$ then we saw above that
$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{f(c)}{g(c)}.$$

Example 6.2.1:
$$\lim_{x\to 1}\left(\frac{x^2+2x+3}{x^2-7}\right) = \frac{6}{-6} = -1.$$

If $f(c) = g(c) = 0$ and f and g are polynomials then we can try to factorise.

Example 6.2.2: (a)
$$\lim_{x\to 1}\left(\frac{x^2-2x+1}{x^2-3x+2}\right) = \lim_{x\to 1}\left(\frac{(x-1)(x-1)}{(x-1)(x-2)}\right) = \frac{0}{-1} = 0.$$
(b)
$$\lim_{x\to 1}\left(\frac{x^3-1}{x^3+2x^2-x-2}\right) = \lim_{x\to 1}\left(\frac{(x-1)(x^2+x+1)}{(x-1)(x^2+3x+2)}\right) = \frac{3}{6} = \frac{1}{2}.$$

Similar methods in reverse may work for other quotients.

Example 6.2.3:
$$\lim_{x\to 4}\left(\frac{\sqrt{x}-2}{x-4}\right) = \lim_{x\to 4}\left(\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}\right)$$
$$= \lim_{x\to 4}\left(\frac{x-4}{(x-4)(\sqrt{x}+2)}\right) = \lim_{x\to 4}\left(\frac{1}{\sqrt{x}+2}\right) = \frac{1}{4}.$$

For $\lim_{x\to 0}$ and $\lim_{x\to\infty}$ of a quotient of polynomials we have the following methods. If we want $\lim_{x\to 0}(\frac{f(x)}{g(x)})$ we first simplify until one or both of f and g has a constant term.

Example 6.2.4: (a)

$$\lim_{x\to 0}\left(\frac{ax^2+bx+c}{Ax^2+Bx+C}\right)=\frac{c}{C} \quad \text{if } C\neq 0.$$

(b)

$$\lim_{x\to 0}\left(\frac{4x^2+3x^3+x^7}{2x^2+x^5}\right)=\lim_{x\to 0}\left(\frac{4+3x+x^5}{2+x^3}\right)=\frac{4}{2}=2$$

(c)

$$\lim_{x\to 0}\left(\frac{3x^2+4x^3+x^4}{x+x^7}\right)=\lim_{x\to 0}\left(\frac{3x+4x^2+x^3}{1+x^6}\right)=\frac{0}{1}=0.$$

To determine $\lim_{x\to\infty}(\frac{f(x)}{g(x)})$ for polynomials f and g we replace x by $\frac{1}{y}$ and use $\lim_{y\to 0}$.

Example 6.2.5:

$$\lim_{x\to\infty}\left(\frac{3+4x+2x^2}{5-x-3x^2}\right)=\lim_{y\to 0}\left(\frac{3+4/y+2/y^2}{5-1/y-3/y^2}\right)$$
$$=\lim_{y\to 0}\left(\frac{3y^2+4y+2}{5y^2-y-3}\right)=-\frac{2}{3}$$

We have one more general rule which can be applied to quotients of arbitrary functions, called l'Hôpital's rule: If $f(c)=g(c)=0$ and $\lim_{x\to c}(\frac{f'(x)}{g'(x)})$ exists then

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right)=\lim_{x\to c}\left(\frac{f'(x)}{g'(x)}\right).$$

This can also be applied to higher derivatives if $f'(c)=g'(c)=0$.

Example 6.2.6: (a) Returning to example 6.2.2(b) we have

$$\lim_{x\to 1}\left(\frac{x^3-1}{x^3+2x^2-x-2}\right)=\lim_{x\to 1}\left(\frac{3x^2}{3x^2+4x-1}\right)=\frac{3}{6}=\frac{1}{2}.$$

(b)

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)=\lim_{x\to 0}\left(\frac{\cos(x)}{1}\right)=\frac{1}{1}=1.$$

This latter example is a standard limit, and should be learnt. Two other standard limits:

$$\lim_{x\to 0}(x^k\ln x)=0 \qquad \text{for } k>0$$

$$\lim_{x\to\infty}(x^k e^{-x})=0 \qquad \text{for } k>0$$

6.3 Convergent series and power series

We say that a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ converges to the number L if the terms in the sequence become arbitrarily close to L for n sufficiently large. In this case we write

$$\lim_{n\to\infty} a_n = L.$$

For example the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ converges to 0 while the sequence $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$ does not converge. As before, this rather vague definition can be made more precise.

Given a sequence of numbers $a_1, a_2, \ldots, a_n, \ldots$ we define the nth partial sum S_n to be

$$S_n=\sum_{i=1}^{n} a_i.$$

We say that the sum $\sum_{i=1}^{\infty} a_i$ converges with sum S if $\lim_{n\to\infty} S_n = S$.

Example 6.3.1: Consider the geometric series with initial value a and constant ratio r. This has nth partial sum

$$S_n=a+ar+\cdots+ar^{n-1}=\frac{a(r^n-1)}{r-1}.$$

The sum $\sum_{k=1}^{\infty} ar^{k-1}$ exists if $\lim_{n\to\infty} S_n$ exists.

If $|r|>1$ then $|r^n|$ gets larger and larger, so there is no limit. But if $|r|<1$ then r^n tends to zero as n tends to infinity. Hence

$$\lim_{n\to\infty}\left(\frac{a(r^n-1)}{r-1}\right)=\frac{a}{1-r}$$

if $|r|<1$. If $|r|=1$ then it can be shown that there is no limit.

In summary, we have

$$\sum_{k=1}^{\infty} ar^{k-1}=\begin{cases}\frac{a}{1-r} & \text{if } |r|<1\\ \text{does not exist} & \text{if } |r|\geq 1.\end{cases}$$

This result must be known — usually it is remembered in the form

$$\sum_{k=0}^{\infty} x^k=\frac{1}{1-x}$$

provided that $|x|<1$.

This is an example of a power series: an infinite sum of increasing powers of a variable x. Typically such series converge for some values of x but not for others.

6.4 Taylor series

Suppose that we have a function which can be differentiated at least n times on some interval containing a point c. We might like to approximate this function by a polynomial (because this is easier to work with), at least in the region near to c.

So what whould be a good approximation to take? It is reasonable to require that the approximation (which we will denote by $p_n(x)$) should agree with $f(x)$ at the point $x=c$, and that the functions should have the same first, second, ..., nth derivatives at the point c. That is, that

$$p_n^{(i)}(c)=f^{(i)}(c)$$

for $0\leq i\leq n$.