Consider the function

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Clearly $p_n(c) = f(c)$, and it is easy to check that

$$p_n{}^{(i)}(c) = f^{(i)}(c)$$

for $1 \leq i \leq n$. Thus $p_n(x)$ approximates $f(x)$ in the desired manner.

We define the Taylor series of f about c to be the infinite sum

$$T(f,c) = \sum_{i \geq 0} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

where $f^{(0)}(x) = f(x)$ and $0! = 1$. When $c = 0$ this is called the Maclaurin series of f.

Example 6.4.1: Find the Maclaurin series of $f(x) = e^x$.

We have $f'(x) = e^x = f''(x) = \ldots$ for all x, and so $f^{(n)}(0) = 1$ for all n. Thus the Maclaurin series for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

As yet we have no guarantee that the Taylor series for a given function will actually converge to equal that function. Indeed, in general it will not converge to the correct value at every value of x. In this course we will not investigate the general problem of convergence, but instead look at some important examples and state (without proof) when they converge.

The following examples should be memorised.

Function	Series	General term	Converges
e^x	$1 + x + \frac{x^2}{2!} + \cdots$	$\frac{x^n}{n!}$	all x
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$\frac{(-1)^n x^{2n}}{(2n)!}$	all x
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\frac{(-1)^{n+1} x^n}{n}$	$-1 < x \leq 1$

Note that we could not give the expansion of $\ln x$ about zero in the above list, but instead about one. Also, when using the formulas for cos and sin we must use radians.

We can also extend the binomial theorem for all real powers p:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p+1-n)}{n!}x^n + \cdots$$

for $-1 < x < 1$.

We can often get one power series by modifying another.

Example 6.4.2: Find a series for $\ln(2+3x)$ and state its region of convergence.

$$\ln(2+3x) = \ln\left(2\left(1+\frac{3x}{2}\right)\right) = \ln 2 + \ln\left(1+\frac{3x}{2}\right).$$

Using the sequence for $\ln(1+u)$ with $u = \frac{3x}{2}$ we have

$$\ln(2+3x) = \ln 2 + \sum_{n \geq 1} \frac{(-1)^{n+1}}{n}\left(\frac{3x}{2}\right)^n.$$

This sequence converges when $-1 < u \leq 1$, i.e. $-\frac{2}{3} < x \leq \frac{2}{3}$.

Example 6.4.3: Find a series for $f(x) = \cos^2 x$.

$$\begin{aligned} \cos^2 x &= \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \\ &= \tfrac{1}{2} + \tfrac{1}{2}\left[1 - \tfrac{(2x)^2}{2!} + \tfrac{(2x)^4}{4!} - \tfrac{(2x)^6}{6!} + \cdots\right] \\ &= 1 - x^2 + \tfrac{x^4}{3} - \tfrac{2x^6}{45} + \cdots \end{aligned}$$

valid for all x.

We often use Taylor series methods to approximate functions close to a value c by a polynomial.

Example 6.4.4: When x is small we have

$$\sin x \approx x \qquad \text{and} \qquad \cos x \approx 1 - \frac{x^2}{2}.$$

These are known as the small angle approximations, and should be known.

Taking this approach to its logical conclusion we see that series expansions are a useful tool for calculating limits of functions.

Example 6.4.5: Calculate

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right).$$

We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and so

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Now

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right) = \lim_{x \to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = 1.$$

Example 6.4.6: Calculate

$$\lim_{x \to 0}\left(\frac{e^{2x} - e^x}{x}\right).$$

We have

$$\begin{aligned} e^{2x} - e^x &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \\ &= x + \frac{3x^2}{2!} + \frac{7x^3}{3!} + \cdots \end{aligned}$$

and so

$$\lim_{x \to 0}\left(\frac{e^{2x} - e^x}{x}\right) = \lim_{x \to 0}\left(1 + \frac{3x}{2!} + \frac{7x^2}{3!} + \cdots\right) = 1.$$

In all of the examples in this section we have concentrated on Maclaurin's series: the special case $c = 0$. This was just to make the calculations easier to write down. For general values of c the methods are the same.

Example 6.4.7: Obtain the Taylor's expansion of $x^2 \ln x$ in powers of $(x-1)$ up to $(x-1)^4$.

Let $f(x) = x^2 \ln x$. Then we want

$$p_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(4)}(1)}{4!}(x-1)^4.$$

You should check that this gives

$$(x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4.$$

6.5 The formal definition of a limit

Our discussion of limits has been somewhat unsatisfactory, as we have not had a rigorous definition of a limit to work with. In this section we will briefly explain how to make this more precise. To give a detailed examination of limits is beyond the scope of this module, so we will restrict ourselves to the definition and some basic examples.

The Real Analysis module in year 2 will return to this topic in much greater detail.

We say that the limit of f as x tends to c is L, written

$$\lim_{x \to c} f(x) = L$$

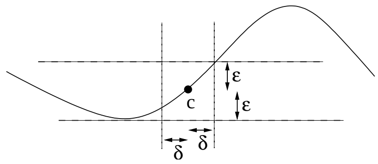
if for all $\epsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

Roughly, this says that $f(x)$ will always be as close to L as we like, provided that we choose x to be sufficiently close to c.

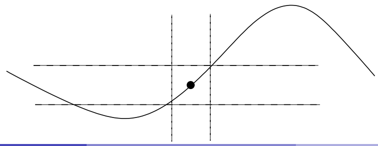
Note that the choice of δ will depend on ϵ.

Sometimes this definition is explained as though we were playing a game. One player chooses a positive number ϵ, and the second player then has to choose a second positive number δ so that $f(x)$ is always within ϵ of L if x is within δ of c. If the second player can always do this, then the function has limit L at c.

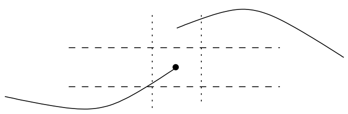
For example, in the following picture Player 1 picks the horizontal strip shown. Then Player 2 can pick the vertical strip as indicated to satisfy the condition.



If Player 1 chooses a narrower horizontal strip as in the next picture, Player 2 can still choose a vertical strip as shown.



For an example where the limit does not exist, consider the function in the following figure.



Here we have indicated a horizontal strip for which no vertical strip will ever guarantee that the curve between the vertical lines will always lie inside the horizontal strip.

Thus this is an example where the limit condition is not satisfied.

These pictures may help us to understand the definition, but they do not help us to apply it. The best way to see how to do this is through some examples.

Example 6.5.1: Show that

$$\lim_{x \to 3} 2x = 6.$$

Here $f(x) = 2x$, $c = 3$, and $L = 6$. Given $\epsilon > 0$, we must find $\delta > 0$ such that

$$|f(x) - 6| < \epsilon$$

whenever $0 < |x - 3| < \delta$. But if we choose $\delta = \epsilon/2$ then for all $0 < |x - 3| < \delta$ we have

$$|f(x) - 6| = |2x - 6| = 2|x - 3| < 2\delta = \epsilon$$

as required.

Example 6.5.2: Show that

$$\lim_{x \to 3} x^2 = 9.$$

Here $f(x) = x^2$, $c = 3$ and $L = 9$. Given $\epsilon > 0$, we must find $\delta > 0$ such that

$$|f(x) - 9| < \epsilon$$

whenever $0 < |x - 3| < \delta$. It is now not as easy to see how to choose δ.

First we suppose that we have picked δ, and see what happens to the equations.

If $|x - 3| < \delta$ then note that

$$|x + 3| = |x - 3 + 6| \leq |x - 3| + 6 \leq \delta + 6.$$

Therefore

$$|x^2 - 9| = |(x - 3)||(x + 3)| \leq \delta(\delta + 6) = \delta^2 + 6\delta.$$

If $\epsilon \geq 1$ then we can choose $\delta = 0.1$, and then

$$|x^2 - 9| < 0.1^2 + 0.6 < \epsilon$$

as required. If $\epsilon < 1$ then $\epsilon^2 < \epsilon$, and we can choose $\delta = \frac{\epsilon}{12}$. Then

$$|x^2 - 9| \leq \delta^2 + 6\delta \leq \frac{\epsilon^2}{144} + \frac{\epsilon}{2} \leq \epsilon(\frac{1}{144} + \frac{1}{2}) < \epsilon$$

as required. Thus we have shown that $\lim_{x \to 3} x^2 = 9$.

Example 6.5.3: Show that

$$\lim_{x\to 0} \sin(\frac{1}{x})$$

does not exist.

Suppose that $\lim_{x\to 0} \sin(\frac{1}{x}) = L$ and take $\epsilon = \frac{1}{4}$. Then if the limit exists we must have that there exists $\delta > 0$ such that

$$\left|\sin\left(\frac{1}{x}\right) - L\right| < \frac{1}{4}$$

whenever $0 < |x| < \delta$. We will show that this is impossible. Let $f(x) = \sin(\frac{1}{x})$.

Suppose that we could find such a $\delta > 0$. Then there exists some $n \in \mathbb{N}$ such that $\frac{2}{(2n+1)\pi} < \delta$. First let $x = \frac{2}{(4n+1)\pi}$. Then

$$\sin\left(\frac{1}{x}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1.$$

Next let $x = -\frac{2}{(4n+1)\pi}$. Then

$$\sin\left(\frac{1}{x}\right) = \sin\left(-2n\pi - \frac{\pi}{2}\right) = -1.$$

Thus we have two values of x (say x_0 and x_1) with $0 < |x| < \delta$ such that $f(x_0) = 1$ and $f(x_1) = -1$. Whatever L is, at least one of $|f(x_0) - L|$ and $|f(x_1) - L|$ must be greater than $\frac{1}{4}$, and so $|f(x) - L|$ is not always less than ϵ. Therefore $f(X)$ has no limit as x tends to 0.

As these examples show, calculating limits can get quite complicated. In this course we will only consider limits in the basic manner described in Sections 6.1–4.

There are similar definitions for limits as x tends to infinity, and for limits of sequences. Using these, all of the results which we have considered in this Chapter can be rigorously derived.

7. Miscellaneous topics

7.1 Proof by induction

Proof by induction can be used when we have a family of propositions $P(n)$ for $n = 1, 2, \ldots$. The idea is to prove that $P(n)$ is true for all values of n by using the following principle:

If there exists m such that both

(i) $P(m)$ is true

(ii) $P(k)$ true implies that $P(k+1)$ is true for all $k \geq m$

then $P(n)$ is true for all $n \geq m$.

As an application of this principle we will prove

Theorem 7.1.1: *For all $n \geq 1$ we have*

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

Proof: Let $P(n)$ be the statement "$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$". We first check $P(1)$:

$$1 = \frac{1}{2} \times 1 \times 2$$

is true.

Now assume $P(k)$ is true for some $k \geq 1$. We must show that this implies that $P(k+1)$ is true.

$P(k)$ states that

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1).$$

The left-hand side of $P(k+1)$ is $1 + 2 + \cdots + k + (k+1)$ which by our assumption equals

$$\frac{1}{2}k(k+1) + k + 1 = \frac{1}{2}(k+1)[k+2]$$

which equals the right-hand side of $P(k+1)$. So $P(k)$ true implies that $P(k+1)$ is true, and the result now follows by induction. □

Note that in proof by induction there are two steps. Although the first step is usually very easy it cannot be omitted.

As another example (with $m \neq 1$) we shall prove

Theorem 7.1.2: *For all $n \geq 4$ we have that $n^2 \leq 2^n$.*

Note that this is false for $n = 3$.

Proof: Let $P(n)$ be "$n^2 \leq 2^n$". Then $P(4)$ is $16 \leq 16$, which is true.

Suppose that $P(k)$ is true for some $k \geq 4$. We want to show that this implies that $(k+1)^2 \leq 2^{k+1}$. We have

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &\leq k^2 + 4k && \text{as } k \geq 1 \\ &\leq k^2 + k^2 && \text{as } k \geq 4 \\ &\leq 2(2^k) && \text{as } P(k) \text{ assumed true} \\ &= 2^{k+1}.\end{aligned}$$

So $P(k)$ true implies that $P(k+1)$ is true, and the result now follows by induction. □

7.2 Sets and elements

A set is a collection of objects. The objects are called elements of the set. We write $x \in A$ for *x is an element of A*, and $x \notin A$ for *x is not an element of A*.

A set may be specified by listing its elements, e.g.

$$A = \{1, 4, 7, 11\}$$

or by stating a common property that defines the set, e.g.

the set of square numbers less than 100

or

$$\{x \in \mathbb{N} : x \text{ is prime}\}.$$

Some important sets are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

A set with a finite number of elements is called finite, otherwise it is called infinite.

If A and B are sets such that every element of A is also an element of B then we say that A is a subset of B, and write $A \subseteq B$.

If $A \subseteq B$ and $B \subseteq A$, i.e. A and B have exactly the same elements, we say that $A = B$.

If A is not a subset of B we write $A \nsubseteq B$.

Example 7.2.1:

(a)
$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}.$$

(b)
$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}.$$

(c)
$$\{1, 2, \{3, 4\}\} \nsubseteq \{1, 2, 3, 4\}.$$

(d)
$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

The set with no elements is called the empty set, denoted by $\emptyset$.

The set of all subsets of A is called the power set of A, written 2^A. If A is finite with n elements then 2^A has 2^n elements.

Example 7.2.2: The power set of

$$\{a, b, \{3, 4\}\}$$

is

$$\{\emptyset, \{a\}, \{b\}, \{\{3, 4\}\}, \{a, b\}, \{a, \{3, 4\}\}, \{b, \{3, 4\}\}, \{a, b, \{3, 4\}\}\}.$$

If $A \subseteq B$ and $A \neq B$ then we may write $A \subset B$ and call A a proper subset of B. In Example 7.2.1 the inclusions in (a) and (d) are proper, but not that in (b). The notation $\subseteq$ and $\subset$ is (deliberately) similar to $\leq$ and $<$ for real numbers.

There are various ways to form new sets from old.

The union $A \cup B$ of two sets is the set of elements x such that $x \in A$ or $x \in B$ (including the possibility that x is in both). That is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The intersection $A \cap B$ of two sets is the set of elements x such that $x \in A$ and $x \in B$. That is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The complement of B in A, $A \backslash B$ is the set of elements $x \in A$ such that $x \notin B$. That is

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

We denote by B' the set

$$\{x : x \notin B\}.$$

Example 7.2.3: Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 3, 4\}$.

$$A \cup B = \{1, 2, 3, 4, 5, 7\},$$

$$A \cap B = \{3\},$$

$$A \backslash B = \{1, 5, 7\}.$$

We can extend the notion of intersections and unions to collections of many sets. If $A_1, A_2, \ldots, A_n$ are sets then we define

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i\}$$

and

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for all } i\}.$$

We can even extend these definitions to infinite collections of sets.

The final pieces of notation we will introduce are the universal and existential quantifiers $\forall$ and $\exists$.

If $p(x)$ is a statement depending on a variable x then $(\forall x)p(x)$ means *$p(x)$ is true for all values of x*, and $(\exists x)p(x)$ means *$p(x)$ is true for at least one value of x*.

Example 7.2.4: If $p(x)$ means *x is an element of B* then $(\forall x \in A)p(x)$ means *every element of A is an element of B*, i.e.

$$A \subseteq B.$$

If $q(x)$ means *x is not an element of B* then $(\exists x \in A)q(x)$ means *some element of A is not an element of B*, i.e.

$$A \nsubseteq B.$$