Ciphers and Number Theory 6

1. Solve each of the following systems of congruences

   (a)
   $$\begin{aligned} x &\equiv 7 \bmod 11 \\ x &\equiv 4 \bmod 13 \\ x &\equiv 8 \bmod 9. \end{aligned}$$

   (b)
   $$\begin{aligned} x &\equiv 4 \bmod 35 \\ x &\equiv 3 \bmod 26 \\ x &\equiv 15 \bmod 19. \end{aligned}$$

2. For each of the following systems of congruences determine whether there is a solution, and solve if possible.

   (a)
   $$\begin{aligned} x &\equiv 1 \bmod 8 \\ x &\equiv 2 \bmod 4 \\ x &\equiv 3 \bmod 17. \end{aligned}$$

   (b)
   $$\begin{aligned} x &\equiv 8 \bmod 15 \\ x &\equiv 3 \bmod 5 \\ x &\equiv 4 \bmod 31. \end{aligned}$$

   (c)
   $$\begin{aligned} x &\equiv 2 \bmod 6 \\ x &\equiv 4 \bmod 8 \\ x &\equiv 1 \bmod 13. \end{aligned}$$

3. Given $n \in \mathbb{N}$, denote by $P(n)$ the set of primes dividing n. Show that the Euler ϕ function satisfies
   $$\phi(n) = n \prod_{p \in P(n)} \left(1 - \frac{1}{p}\right).$$

4. Calculate the value of $\phi(168)$.