## 1.11 The central rate of mortality

From above, we have seen that $q_x$ represents the probability that a life of exact age $x$ dies before reaching exact age $(x+1)$.

Then, $q_x$ is often referred to as the *initial* rate of mortality at exact age $x$.

An alternative definition of the rate of mortality is often used in demography.

We define the *central* rate of mortality at exact age $x$, denoted by $m_x$, as follows:

$$m_x = \frac{\int_0^1 l_{x+t}\mu_{x+t}\,dt}{\int_0^1 l_{x+t}\,dt} = \frac{\int_0^1 {}_t p_x \mu_{x+t}\,dt}{\int_0^1 {}_t p_x\,dt} = \frac{q_x}{\int_0^1 {}_t p_x\,dt} \qquad (1.11.1)$$

In practice, the central rate of mortality $m_x$ represents a weighted average of the force of mortality applying over the year of age $x$ to $(x+1)$, and can be thought as the probability that a life alive between ages $x$ and $(x+1)$ dies before attaining exact age $(x+1)$.

The importance of the central rate of mortality $m_x$ arose because, historically, it was easier for actuaries to estimate this quantity from the observed data than either the initial rate of mortality, $q_x$, or the force of mortality, $\mu_x$.

## 1.12 Expectation of life

### 1.12.1 Complete expectation of life

From Section 1.2, the random variable $T_x$ represents the complete future lifetime for a life of exact age $x$.

Then, the expected value of the random variable $T_x$, denoted by $\overset{\circ}{e}_x$, is the *complete expectation of life* for a life of age $x$.

From (1.5.4), the probability density function of the random variable $T_x$ is given by:

$$f_x(t) = {}_t p_x \mu_{x+t} \qquad \text{for } t \geq 0$$

Note that $\overset{\circ}{e}_x$ is the expected future lifetime after age $x$, so that, for a life of exact age $x$, the expected age at death is $\left( x + \overset{\circ}{e}_x \right)$.

Now, by definition, we have:

$$\overset{\circ}{e}_x = E(T_x) = \int_0^\infty t \times f_x(t)dt = \int_0^\infty t \times {}_t p_x \mu_{x+t} dt \tag{1.12.1}$$

Then, from (1.5.5), we have $\frac{\partial}{\partial t} {}_t p_x = -{}_t p_x \mu_{x+t}$, and using integration by parts, we obtain:

$$\begin{aligned} \overset{\circ}{e}_x &= \int_0^\infty t \times {}_t p_x \mu_{x+t} dt \\ &= \int_0^\infty t \times \left( -\frac{\partial}{\partial t} {}_t p_x \right) dt \\ &= \left[ t \times (-{}_t p_x) \right]_{t=0}^{t=\infty} - \int_0^\infty (-{}_t p_x) dt \\ &= \int_0^\infty {}_t p_x dt \end{aligned} \tag{1.12.2}$$

**Example 1.12.1**

In a particular survival model, we have:

$$\mu_x = \frac{0.01}{1-0.01x} \qquad \text{for } 0 \le x < 100$$

Find the complete expectation of life at exact age 20.

**Solution**

Firstly, we must find ${}_t p_{20}$, the survival function for a life of exact age 20.

From (1.7.1), we have:

$$\begin{aligned}
{}_t p_{20} &= \exp\left( - \int_{20}^{20+t} \mu_s ds \right) \\
&= \exp\left( - \int_{20}^{20+t} \frac{0.01}{1-0.01s} ds \right) \\
&= \exp\left( - \left[ -\ln(1-0.01s) \right]_{s=20}^{s=20+t} \right) \\
&= \frac{1-0.01\times(20+t)}{1-0.01\times 20} \\
&= 1 - \frac{0.01t}{1-0.01\times 20} \\
&= 1 - \frac{t}{80}
\end{aligned}$$

As the limiting age in the survival model is 100, the complete future lifetime for a life of exact age 20 must be less than 80 years.

Then, from (1.8.2), we have:

$$\begin{aligned}
\overset{\circ}{e}_{20} &= \int_0^{100\text{-}20} {}_t p_{20} dt \\
&= \int_0^{80} \left( 1 - \frac{t}{80} \right) dt \\
&= \left[ t - \frac{t^2}{40} \right]_{t=0}^{t=80} \\
&= 40
\end{aligned}$$

Thus, the complete expectation of life for a life of exact age 20 is 40 years.

□

The complete expectation of life, typically for a new-born life, is often used to compare the general level of health in different populations.

For example, the life expectancy for a new-born male life in different countries is:

| **Country** | **Life expectancy** |
|---|---|
| Japan | 77.5 |
| United Kingdom | 75.0 |
| Germany | 74.3 |
| United States | 74.2 |
| Mexico | 68.5 |
| Russia | 62.0 |
| South Africa | 50.4 |
| Zimbabwe | 39.2 |

*Source: US Bureau of the Census, International data base, June 2000*

Also, using integration by parts, it can also be seen that:

$$E\left(T_x^2\right) = \int_0^\infty t^2 \times {}_t p_x \mu_{x+t} dt = 2 \times \int_0^\infty t \times {}_t p_x dt$$

Thus, the variance of the complete future lifetime for a life of exact age $x$ is given by:

$$\text{var}(T_x) = E\left(T_x^2\right) - \left[E(T_x)\right]^2 = 2 \times \int_0^\infty t \times {}_t p_x dt - \left(\int_0^\infty {}_t p_x dt\right)^2 \qquad (1.12.3)$$

### 1.12.2 Curtate expectation of life

The random variable $K_x$ is used to represent the curtate future lifetime for a life of exact age $x$ (i.e. the number of *complete* years lived after age $x$).

Then, the random variable $K_x$ is the integer part of the complete future lifetime, $T_x$.

Clearly, $K_x$ is a *discrete* random variable taking values in the state space $J = 0,1,2,\ldots$.

We can use the distribution function of $T_x$, denoted by $F_x(t)$, to derive the probability distribution function of $K_x$ as follows:

$$
\begin{aligned}
\Pr(K_x = k) &= \Pr(k \le T_x < k+1) \\
&= F_x(k+1) - F_x(k) \\
&= (1 - {}_{k+1}p_x) - (1 - {}_k p_x) \\
&= {}_k p_x - {}_{k+1} p_x \qquad (1.12.4) \\
&= {}_k p_x \times (1 - p_{x+1}) \\
&= {}_k p_x \times q_{x+k} \\
&= \frac{d_{x+k}}{l_x}
\end{aligned}
$$

This result is intuitive.

If the random variable $K_x$ takes the value $k$, then a life of exact age $x$ must live for $k$ complete years after age $x$. Therefore, the life must die in the year of age $(x+k)$ to $(x+k+1)$.

From above, we have seen that, for a life of exact age $x$, the probability of death in the year of age $(x+k)$ to $(x+k+1)$ is ${}_{k|}q_x = \dfrac{d_{x+k}}{l_x} = \Pr(K_x = k)$.

Now, the expected value of the random variable $K_x$, denoted by $e_x$, is known as the *curtate expectation of life* for a life of age $x$.

Thus, we have:

$$\begin{aligned} e_x = E(K_x) &= \sum_{k=0}^{\infty} k \times \Pr(K_x = k) \\ &= \sum_{k=0}^{\infty} k \times \frac{d_{x+k}}{l_x} \\ &= \frac{d_{x+1} + 2 \times d_{x+2} + 3 \times d_{x+3} + \ldots}{l_x} \\ &= \frac{(l_{x+1} - l_{x+2}) + 2 \times (l_{x+2} - l_{x+3}) + 3 \times (l_{x+3} - l_{x+4}) + \ldots}{l_x} \\ &= \frac{l_{x+1} + l_{x+2} + l_{x+3} + \ldots}{l_x} \\ &= \sum_{k=1}^{\infty} {}_k p_x \end{aligned} \tag{1.12.5}$$

If required, we can also calculate the variance of the curtate future lifetime as follows:

$$\operatorname{var}(K_x) = E(K_x^2) - [E(K_x)]^2 = \sum_{k=0}^{\infty} k^2 \times \frac{d_{x+k}}{l_x} - (e_x)^2 \tag{1.12.6}$$

### 1.12.3 Relationship between $\overset{\circ}{e}_x$ and $e_x$

Assuming that the function ${}_t p_x$ is linear between integer ages, we have:

$$
\begin{aligned}
\overset{\circ}{e}_x &= \int_0^\infty {}_t p_x dt \\
&\approx \frac{1}{2}\times\left({}_0 p_x + {}_1 p_x\right) + \frac{1}{2}\times\left({}_1 p_x + {}_2 p_x\right) + \ldots \\
&= \frac{1}{2}\times {}_0 p_x + \sum_{k=1}^{\infty} {}_k p_x \\
&= e_x + \frac{1}{2}
\end{aligned}
\qquad (1.12.7)
$$

Thus, the *complete* expectation of life at age $x$ is **approximately** equal to the *curtate* expectation of life plus one-half of a year.

This is equivalent to the assumption that lives dying in the year of age $(x+k)$ to $(x+k+1)$ do so, on average, half-way through the year at age $(x+k+\frac{1}{2})$.

This assumption is known as the *uniform distribution of death* assumption.

It should be noted that, whilst the curtate future lifetime $K_x$ is equal to the integer part of the complete future lifetime $T_x$, the curtate expectation of life $e_x$ is **not** equal to the integer part of the complete expectation of life $\overset{\circ}{e}_x$.

## 1.13 Interpolation for the life table

As discussed previously, it is common for the standard life table functions such as $l_x$, $q_x$ or $\mu_x$ to be tabulated at integer ages only.

However, the actuary may be required to calculate probabilities involving non-integer ages or durations.

Then, given a life table $\{l_x : x = \alpha, \alpha+1, \ldots, \omega\}$ specified only at integer ages, how can we approximate the values of $l_{x+t}$ (where $x$ is an integer and $0 < t < 1$)?

We consider three possible approaches.

### 1.13.1 Uniform distribution of deaths (UDD)

In this case, we assume that any deaths over the year of age $x$ to $(x+1)$ occur uniformly over the year.

This is equivalent to the assumption that the function $l_{x+t}$ is linear over the interval $(x, x+1)$.

Thus, for $0 < t < 1$, we have $l_{x+t} = (1-t) \times l_x + t \times l_{x+1} = l_x - t \times (l_x - l_{x+1}) = l_x - t \times d_x$.

Hence, under the UDD assumption, dividing both sides by $l_x$ gives:

$$ {}_t p_x = 1 - t \times q_x \Rightarrow {}_t q_x = 1 - {}_t p_x = t \times q_x $$

Then, under the assumption that the function $l_{x+t}$ is linear over the interval $(x, x+1)$, we have:

$$ {}_t q_x = \int_0^t {}_s p_x \mu_{x+s} ds = t \times q_x \qquad (1.13.1) $$

Thus, as the function $q_x$ is tabulated, we can estimate the probability ${}_t q_x$ for any non-integer durations $t$.

Note that, differentiating both sides of this expression with respect to $t$, we obtain:

$$ q_x = \frac{d}{dt}\left( \int_0^t {}_s p_x \mu_{x+s} ds \right) = {}_t p_x \mu_{x+t} = f_x(t) \qquad \text{for } 0 < t < 1 $$

Thus, under the assumption that the function $l_{x+t}$ is linear over the interval $(x, x+1)$, the distribution function of the complete future lifetime, $T_x$, is constant for $0 < t < 1$.

Hence, deaths are *uniformly distributed* over the year of age $x$ to $(x+1)$.

We can extend this approach when both the age and the duration are non-integer values, so as to enable us to estimate the probability ${}_{t-s}q_{x+s}$ where $x$ is an integer and $0 < s < t < 1$).

In this case, we can write ${}_t p_x = {}_s p_x \times {}_{t-s}p_{x+s} \Rightarrow {}_{t-s}p_{x+s} = \frac{{}_t p_x}{{}_s p_x}$.

Thus, we can express ${}_{t-s}q_{x+s}$ as ${}_{t-s}q_{x+s} = 1 - {}_{t-s}p_{x+s} = 1 - \frac{{}_t p_x}{{}_s p_x} = 1 - \frac{1 - {}_t q_x}{1 - {}_s q_x}$.

And, using the UDD assumption, we have:

$$ {}_{t-s}q_{x+s} = 1 - \frac{1 - t \times q_x}{1 - s \times q_x} = \frac{(t-s) \times q_x}{1 - s \times q_x} \quad \text{for } 0 < s < t < 1 \qquad (1.13.2) $$

Also, using the UDD assumption, we can express the central rate of mortality at age $x$, $m_x$, in two different ways:

(i) If the function ${}_t p_x$ is linear for $0 \le t \le 1$, then we have $\int_0^1 {}_t p_x dt = {}_{\frac{1}{2}}p_x$ (i.e. the value of the function ${}_t p_x$ at the mid-point of the interval). Thus, we have:

$$ m_x = \frac{q_x}{\int_0^1 {}_t p_x dt} = \frac{q_x}{{}_{\frac{1}{2}}p_x} = \frac{q_x}{1 - {}_{\frac{1}{2}}q_x} = \frac{q_x}{1 - \frac{1}{2} \times q_x} \qquad (1.13.3) $$

(ii) If the function $f_x(t) = {}_t p_x \mu_{x+t}$ is constant for $0 \le t \le 1$, then we can put $t = \frac{1}{2}$ giving $f_x(t) = {}_{\frac{1}{2}}p_x \mu_{x+\frac{1}{2}}$ for all $t \in [0,1]$. Thus, we have:

$$ m_x = \frac{\int_0^1 {}_t p_x \mu_{x+t} dt}{\int_0^1 {}_t p_x dt} = \frac{\int_0^1 {}_{\frac{1}{2}}p_x \mu_{x+\frac{1}{2}} dt}{{}_{\frac{1}{2}}p_x} = \frac{{}_{\frac{1}{2}}p_x \mu_{x+\frac{1}{2}} \times \int_0^1 dt}{{}_{\frac{1}{2}}p_x} = \mu_{x+\frac{1}{2}} \qquad (1.13.4) $$

### 1.13.2 Constant force of mortality

In this case, we assume that the function $\mu_{x+t}$ is constant over the year of age $x$ to $(x+1)$.

i.e. for integer $x$ and $0 < t < 1$, we have $\mu_{x+t} = \mu = \text{constant}$

Note that, in general, the value of $\mu$, the constant force of mortality assumed over the year of age $x$ to $(x+1)$, will not be equal to either of the tabulated values $\mu_x$ or $\mu_{x+1}$.

Under the assumption of a constant force of mortality between integer ages, we find the value of the constant $\mu$ using:

$$p_x = \exp\left(-\int_0^1 \mu_{x+t}dt\right) = e^{-\mu} \Rightarrow \mu = -\ln(p_x) \tag{1.13.5}$$

Then, for $0 < t < 1$, we have:

$$\begin{aligned} {}_tq_x &= 1 - {}_tp_x \\ &= 1 - \exp\left(-\int_0^t \mu_{x+s}ds\right) \\ &= 1 - \exp\left(-\int_0^t \mu ds\right) \\ &= 1 - e^{-t\mu} \end{aligned} \tag{1.13.6}$$

Similarly, when we have a non-integer age and duration, we estimate the probability ${}_{t-s}q_{x+s}$, for $0 < s < t < 1$, as follows:

$$\begin{aligned} {}_{t-s}q_{x+s} &= 1 - {}_{t-s}p_{x+s} \\ &= 1 - \exp\left(-\int_s^t \mu_{x+r}dr\right) \\ &= 1 - \exp\left(-\int_s^t \mu dr\right) \\ &= 1 - e^{-(t-s)\mu} \end{aligned} \tag{1.13.7}$$

**Example 1.9.1**

Given $p_{90} = 0.75$, calculate ${}_{\frac{1}{12}}q_{90}$ and ${}_{\frac{1}{12}}q_{90\frac{11}{12}}$ assuming:

(a) a uniform distribution of deaths between integer ages, and

(b) a constant force of mortality between integer ages.

**Solution**

(a) Uniform distribution of deaths

From (1.13.1), we have:

$$ {}_{\frac{1}{12}}q_{90} = \frac{1}{12} \times q_{90} = \frac{1}{12} \times (1 - p_{90}) = \frac{1}{12} \times (1 - 0.25) = 0.020833 $$

Also, from (1.13.2) with $s = \frac{11}{12}$ and $t = 1$, we have:

$$ {}_{\frac{1}{12}}q_{90\frac{11}{12}} = \frac{\left(1 - \frac{11}{12}\right) \times q_{90}}{1 - \frac{11}{12} \times q_{90}} = \frac{\frac{1}{12} \times 0.25}{1 - \frac{11}{12} \times 0.25} = 0.027027 $$

(b) Constant force of mortality

First, we must find the value of $\mu$, the constant force of mortality over the year of age $(90,91)$.

Then, from (1.13.5), we have $\mu = -\ln(p_{90}) = -\ln(0.75) = 0.287682$.

From (1.13.6), we have:

$$ {}_{\frac{1}{12}}q_{90} = 1 - e^{-\frac{1}{12} \times \mu} = 0.023688 $$

Also, from (1.13.7) with $s = \frac{11}{12}$ and $t = 1$, we have:

$$ {}_{\frac{1}{12}}q_{90\frac{11}{12}} = 1 - e^{-\left(1-\frac{11}{12}\right)\times\mu} = 1 - e^{-\frac{1}{12}\times\mu} = 0.023688 $$

□

Note that, under the constant force of mortality assumption, the central rate of mortality at age $x$, $m_x$, is given by:

$$ m_x = \frac{\int_0^1 {}_t p_x \mu_{x+t}\, dt}{\int_0^1 {}_t p_x\, dt} = \frac{\mu \times \int_0^1 {}_t p_x\, dt}{\int_0^1 {}_t p_x\, dt} = \mu \qquad (1.13.8) $$

### 1.13.3 The Balducci assumption

The Italian actuary Balducci proposed an alternative approach for estimating probabilities at non-integer ages and durations.

The approach is based on the traditional actuarial method of constructing a life table, which will be considered in more detail later.

The assumption is that the function $l_{x+t}$ is in form hyperbolic between integer ages.

Note that, as mentioned previously, the UDD assumption implies that the function $l_{x+t}$ is linear between integer ages, whereas the constant force of mortality assumption implies that the function $l_{x+t}$ is exponential between integer ages.

Then, for any integer $x$ and $0 < t < 1$, using hyperbolic interpolation, we have $\frac{1}{l_{x+t}} = \frac{1-t}{l_x} + \frac{t}{l_{x+1}}$.

Thus, for $0 < t < 1$, we can write:

$$\frac{1}{l_{x+t}}=\frac{(1-t)\times l_{x+1}+t\times l_x}{l_x\times l_{x+1}}\Rightarrow\frac{l_{x+1}}{l_{x+t}}=\frac{l_x-(1-t)\times(l_x-l_{x+1})}{l_x}$$

Hence, the Balducci assumption is usually expressed as:

$${}_{1-t}p_{x+t}=1-(1-t)\times\frac{d_x}{l_x}=1-(1-t)\times q_x\Rightarrow{}_{1-t}q_{x+t}=1-{}_{1-t}p_{x+t}=(1-t)\times q_x \qquad (1.13.9)$$

Now, using the Balducci assumption, we have:

$$p_x={}_tp_x\times{}_{1-t}p_{x+t}\Rightarrow{}_tp_x=\frac{p_x}{{}_{1-t}p_{x+t}}\Rightarrow{}_tq_x=1-\frac{p_x}{{}_{1-t}p_{x+t}}=1-\frac{1-q_x}{1-(1-t)\times q_x}$$

Hence, for integer age $x$ and $0<t<1$, the Balducci assumption gives:

$${}_tq_x=1-\frac{1-q_x}{1-(1-t)\times q_x}=\frac{t\times q_x}{1-(1-t)\times q_x} \qquad (1.13.10)$$

By definition, the assumption of a constant force of mortality assumes that the function $\mu_{x+t}$ is constant over the year of age $x$ to $(x+1)$.

Now, combining (1.2.5) and (1.3.4), we have $\mu_{x+t}=-\frac{1}{l_{x+t}}\times\frac{d}{dt}(l_{x+t})$.

For the UDD assumption, we have $l_{x+t}=l_x-t\times(l_x-l_{x+1})\Rightarrow\frac{d}{dt}(l_{x+t})=-(l_x-l_{x+1})$.

Thus, using the UDD assumption, we can express the force of mortality at age $(x+t)$ as:

$$\mu_{x+t}=\frac{l_x-l_{x+1}}{l_x-t\times(l_x-l_{x+1})}=\frac{q_x}{1-t\times q_x} \qquad (1.13.11)$$

Thus, under the UDD assumption, the force of mortality is an *increasing* function over the year of age $x$ to $(x+1)$.

This result can be explained by general reasoning.

Consider a group of lives who die at a uniform rate over a given year.

Then, to maintain a constant number of deaths over the year, the force of mortality must increase to offset the fact that the number of survivors is decreasing over time.

Also, this result is intuitive and consistent with the expected pattern for the force of mortality for human populations (i.e. we expect the force of mortality to be an increasing function of age).

Similarly, for the Balducci assumption, it can shown that the force of mortality at age $(x+t)$ is given by:

$$\mu_{x+t} = \frac{q_x}{1-(1-t)\times q_x} \qquad (1.13.12)$$

Thus, under the Balducci assumption, the force of mortality is a *decreasing* function over the year of age $x$ to $(x+1)$.

This result is counter-intuitive and inconsistent with the expected pattern for the force of mortality for human populations.

However, as mentioned previously, the assumption is useful in the traditional actuarial method of constructing a life table (and will be considered further later).

## 1.14 Simple analytical laws of mortality

It may be possible to postulate an analytical form for one of the standard life table functions such as $l_x$, $q_x$ or $\mu_x$.

Such an approach simplifies the construction of a suitable life table from crude mortality data (as the number of parameters required to be estimated is substantially reduced), but the mathematical formulae used must be representative of the actual underlying mortality experience (and is now considered unlikely that a simple analytical expression can be proposed that will adequately represent human mortality over a large range of ages).

However, before the recent advancements in computing speed and storage capacity, this approach was reasonably common and we now consider some of better-known laws of mortality proposed.

### 1.14.1 De Moivre's Law

De Moivre's Law was proposed in 1729 and states that, for all ages $x$ such that $0 \le x < \omega$, we have:

$$\mu_x = \frac{1}{\omega - x} \tag{1.14.1}$$

Thus, as expected, the force of mortality is an increasing function of age.

Then, we can derive the survival function as follows:

$$\begin{aligned} {}_t p_x &= \exp\left( -\int_x^{x+t} \mu_s \, ds \right) \\ &= \exp\left( -\int_x^{x+t} \frac{1}{\omega - s} \, ds \right) \\ &= \exp\left( \left[ \ln(\omega - s) \right]_{s=x}^{s=x+t} \right) \\ &= \frac{\omega - (x+t)}{\omega - x} \end{aligned} \tag{1.14.2}$$

### 1.14.2 Gompertz' Law

Gompertz' Law was proposed in 1829 and was based on the observation that, over a large range of ages, the function $\mu_x$ is log-linear.

Thus, for all ages $x \geq 0$, we have:

$$\mu_x = Bc^x \tag{1.14.3}$$

Then, assuming that the underlying force of mortality follows Gompertz' Law, the parameter values $B$ and $c$ can be determined given the value of the force of mortality at any two ages.

To ensure that the force of mortality is a non-negative increasing function of age, we require that the parameter values $B$ and $c$ are such that $B > 0$ and $c > 1$.

We can derive the survival function as follows:

$$\begin{aligned}
{}_t p_x &= \exp\left(-\int_0^t \mu_{x+s} ds\right) \\
&= \exp\left(-\int_0^t Bc^{x+s} ds\right) \\
&= \exp\left(-\int_0^t Bc^x e^{s\ln(c)} ds\right) \\
&= \exp\left(\frac{-B}{\ln(c)} c^x \left[e^{s\ln(c)}\right]_{s=0}^{s=t}\right) \\
&= \exp\left(\frac{-B}{\ln(c)} c^x \left(c^t - 1\right)\right)
\end{aligned}$$

Now, if we define the parameter $g$ such that $g = \exp\left(\frac{-B}{\ln(c)}\right)$, then we can express the survival function as:

$${}_t p_x = \exp\left[\ln(g) c^x \left(c^t - 1\right)\right] = g^{c^x\left(c^t - 1\right)} \tag{1.14.4}$$

In practice, Gompertz' Law is often found to be a reasonable approximation for the force of mortality at older ages.

### 1.14.3 Makeham's Law

Makeham's Law was proposed in 1860, and incorporated the addition of a constant term in the expression for the force of mortality.

The rationale behind this is that an age-independent allowance is required for the incidence of accidental deaths.

Thus, for all ages $x \geq 0$, we have:

$$\mu_x = A + Bc^x \tag{1.14.5}$$

Then, assuming that the underlying force of mortality follows Makeham's Law, the parameter values $A$, $B$ and $c$ can be determined given the value of the force of mortality at any three ages.

To ensure that the force of mortality is a non-negative increasing function of age, we require that the parameter values $A$, $B$ and $c$ are such that $A \geq -B$, $B > 0$ and $c > 1$.

We can derive the survival function using the same approach adopted above for Gompertz' Law to obtain:

$$_t p_x = s^t g^{c^x\left(c^t - 1\right)} \tag{1.14.6}$$

where $s = \exp(-A)$ and $g = \exp\left(\dfrac{-B}{\ln(c)}\right)$.

**Example 1.14.1**

A survival model is assumed to follow Makeham's Law for the force of mortality at age $x$, $\mu_x$.

Then, given that ${}_5p_{70} = 0.70$, ${}_5p_{80} = 0.40$ and ${}_5p_{90} = 0.15$, find the values of the parameters $A$, $B$ and $c$.

Hence, or otherwise, find the probability that a life of exact age 50 will die between exact ages 55 and 65.

**Solution**

From (1.14.6), we have:

$$
\begin{aligned}
{}_5p_{70} &= s^5 g^{c^{70}(c^5-1)} = 0.70 \quad --- \quad (1) \\
{}_5p_{80} &= s^5 g^{c^{80}(c^5-1)} = 0.40 \quad --- \quad (2) \\
{}_5p_{90} &= s^5 g^{c^{90}(c^5-1)} = 0.15 \quad --- \quad (3)
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
\frac{(2)}{(1)} &\Rightarrow g^{c^{70}(c^{10}-1)(c^5-1)} = \frac{0.40}{0.70} \quad --- \quad (4) \\
\frac{(3)}{(2)} &\Rightarrow g^{c^{80}(c^{10}-1)(c^5-1)} = \frac{0.15}{0.40} \quad --- \quad (5)
\end{aligned}
$$

Then, taking logarithms of (4) and (5) gives:

$$
\begin{aligned}
c^{70}(c^{10}-1)(c^5-1)\times\ln(g) &= \ln\left(\frac{0.40}{0.70}\right) \quad --- \quad (6) \\
c^{80}(c^{10}-1)(c^5-1)\times\ln(g) &= \ln\left(\frac{0.15}{0.40}\right) \quad --- \quad (7)
\end{aligned}
$$

And, dividing $(7)$ by $(6)$ gives:

$$
c^{10} = \frac{\ln\left(\frac{0.15}{0.40}\right)}{\ln\left(\frac{0.40}{0.70}\right)} \Rightarrow c = 1.057719
$$

Then, from $(4)$, we have $\ln(g) = \dfrac{\ln\left(\dfrac{0.40}{0.70}\right)}{c^{70}\left(c^{10}-1\right)\left(c^{5}-1\right)} = -0.045181 \Rightarrow g = 0.955824$.

Now, from (1.14.6), we have $g = \exp\left(\dfrac{-B}{\ln(c)}\right) \Rightarrow B = 0.002535$.

And, taking the logarithm of $(1)$, gives:

$$\ln(0.70) = 5 \times \ln(s) + c^{70}\left(c^{5}-1\right) \times \ln(g) \Rightarrow \ln(s) = 0.077364$$

From (1.14.6), we have $s = \exp(-A) \Rightarrow A = -0.077364$.

Thus, the force of mortality at age $x$ is given by $\mu_x = -0.077364 + 0.002535 \times (1.057719)^x$.

□

## 1.15 The select mortality table

Before being accepted for life assurance cover, potential policyholders are often required to undergo a medical examination to satisfy the insurer that they are in a 'reasonable' level of health.

Lives who fail to satisfy the requirements laid down by the insurance company will often be refused cover (or required to pay a higher premium for the same level of cover).

As a result of this filtering, lives who have recently been accepted for cover can be expected to be in better health (and, thus, experience lighter mortality) than the general population at the same age.

This effect is known as *selection* (i.e. the process of choosing lives for membership of a defined group, rather than random sampling).

However, as the duration since selection increases, the extent of the lighter mortality experienced by the select group of lives can be expected to reduce (as previously healthy individuals are exposed to the same medical conditions as the general population).

In practice, *select* lives are often assumed to experience lighter mortality for a period of, say, $s$ years (known as the *select period*). However, once the duration since selection exceeds the select period, the lives are assumed to experience the ultimate mortality rates appropriate for the general population at the same age.

Thus, we now consider the construction and application of a *select* life table, where mortality varies by age **and** duration since selection.

The A1967-70 mortality table uses a select period of two years, so that select lives are assumed to experience lighter mortality for the first two years after selection (before reverting to the mortality experience of the general population, as represented by the ultimate portion of the table).

However, the a(55) table uses a select period of one year

And, the ELT No. 15 – Males table is an ultimate life table only (i.e. there is no select period). This is commonly referred to as an *aggregate* mortality table.

Examples of selection include:

(a) temporary initial selection

- that exercised by a life assurance company in deciding whether or not to accept a person for life assurance cover
- selection takes place by producing satisfactory medical evidence
- known as *underwriting*

(b) self selection

- that exercised by lives when choosing to purchase an annuity (i.e. exchanging a capital sum for the receipt of an income for life)

These are examples of *positive selection*, where the select lives are likely to experience *lower* mortality rates than the general (or *ultimate*) population of the same age for a specified duration since selection only.

However, a life retiring early on grounds of ill-health is likely to experience higher mortality than the ultimate population of the same age. This is an example of *negative selection*.

### 1.15.1 Select, ultimate and aggregate mortality rates

Most select life tables are constructed to explore the effect of temporary initial selection (i.e. where selected lives experience *lighter* mortality than the general population studied for a specified duration since selection).

Suppose that the select period is $s$ years.

Consider a life who is currently of exact age $(x+r)$, and who was selected at age $x$.

Thus, the duration since selection is $r$ years.

Now, if $r < s$, then we expect the life to experience lower mortality than the ultimate population at the same age and we define the *select* mortality rate at age $(x+r)$ as follows:

$$q_{[x]+r} = \Pr\left[\text{life now aged } (x+r)\text{, who joined select group at age } x\text{, dies before age } (x+r+1)\right]$$

Note that $[x]$ is used to denote the age at selection and $r$ is the duration since selection, so that the current age of the life is $(x+r)$.

Thus, as the life is expected to experience lower mortality than an ultimate life of the same age, we have:

$$q_{[x]+r} < q_{x+r} \quad \text{for } r < s \qquad (1.15.1)$$

And, as before, we have $p_{[x]+r} = 1 - q_{[x]+r}$.

Similarly, consider another life who is also currently of exact age $(x+r)$, but who was selected at age $(x+1)$.

Thus, in this case, the duration since selection is $(r-1)$ years.

We define the select mortality rate at age $(x+r)$ for this life as follows:

$$q_{[x+1]+(r-1)} = \Pr[\text{life now aged } (x+r)\text{, who joined select group at age } (x+1)\text{, dies before age } (x+r+1)]$$

Note that, in this case, $[x+1]$ is used to denote the age at selection and $(r-1)$ is the duration since selection, so that the current age of the life is also $(x+r)$.

However, as this life has been selected more recently, we would expect this life to experience lighter mortality over the year of age $(x+r)$ to $(x+r+1)$ than the life selected at age $x$.

Thus, we have:

$$q_{[x+1]+(r-1)} < q_{[x]+r} \qquad \text{for } r < s \qquad (1.15.2)$$

However, if $r \geq s$, then we expect lives of the same age who were selected $s$ or more years previously have the **same** rates of mortality, *regardless of age at selection*.

In this case, all lives selected $s$ or more years previously will experience the rates of mortality of the ultimate population at the same age.

For the A1967-70 life table, the select period is 2 years.

Then, for lives of age $(x+2)$ and select durations of 2 years or more, we have:

$$q_{[x]+2} = q_{[x-1]+3} = q_{[x-2]+4} = \ldots = q_{x+2} \qquad (1.15.3)$$

However, for select durations of less than two years, we have:

$$q_{[x+1]+1} < q_{x+2} \text{ and } q_{[x+2]} < q_{[x+1]+1} < q_{x+2} \tag{1.15.4}$$

Select mortality table function are generally displayed in the form of an array.

An extract from the A1967-70 table is shown below.

| age $[x]$ | $q_{[x]}$ | $q_{[x]+1}$ | $q_{x+2}$ | age $x+2$ |
|---|---|---|---|---|
| 60 | 0.00669904 | 0.00970168 | 0.01774972 | 62 |
| 61 | 0.00723057 | 0.01055365 | 0.01965464 | 63 |
| 62 | 0.00779397 | 0.01146756 | 0.02174310 | 64 |
| 63 | 0.00839065 | 0.01244719 | 0.02403101 | 65 |
| 64 | 0.00902209 | 0.01349653 | 0.02653550 | 66 |

The convention is that each row represents how mortality rates change as duration since selection increases.

Thus, for a life selected at age 60, denoted by $[60]$, the rate of mortality in the year of age 60 to 61 is $q_{[60]}$ and the rate of mortality in the year of age 61 to 62 is $q_{[60]+1}$.

However, two years after selection, the lighter mortality experienced as a result of selection is assumed to wear off, and the rate of mortality experienced in the year of age 62 to 63 is simply that of the ultimate population at the same age, $q_{62}$.

Thereafter, the life is assumed to be an ultimate life and so, for any duration since selection $r \geq 2$, the rate of mortality experienced in the year of age $(x+r)$ to $(x+r+1)$ is $q_{x+r}$.

Also, the rates displayed on the upwards diagonal represent the rate of mortality experienced by lives of the same age **but** with a different duration since selection.

Thus, the rates of mortality $q_{[62]}$, $q_{[61]+1}$ and $q_{62}$ all apply to the year of age 62 to 63, but the duration since selection is zero years, one year and two (or more) years respectively.

As expected, we can see that $q_{[62]} < q_{[61]+1} < q_{62}$, so that lives selected more recently can be expected to experience lighter mortality rates over the particular year of age.

Note the large difference that selection can make to mortality experience.

For example, for a life of age 62, the rate of mortality for a newly-selected life, given by $q_{[62]} = 0.00723057$, is less than half that of an ultimate life of the same age, given by $q_{62} = 0.01774972$.

From inspection of the full table, this effect becomes more pronounced as the age at selection increases.

### 1.15.2 Constructing a select mortality table

As discussed previously, a life table is a convenient method of summarising the information contained within the survival model.

The only difference now is that the survival probabilities depend not only on age but also on duration since selection.

Given the select mortality rates, $q_{[x]+r}$, for all possible ages at selection $[x]$ and durations since selection $r < s$ (where $s$ is the chosen select period) **and** the ultimate mortality rates, $q_{x+s}$, for all possible ultimate ages $(x+s)$, a life table representing the select and ultimate experience can be constructed.

Note that, in practice, the length of the select period would usually be determined from the observed data by finding the duration since selection after which the mortality experience did not appear to differ significantly from other lives of the same age *but* with a lower age at selection.

Then, the ultimate mortality rates would be based on the grouped experience of all lives of the same age after the end of the chosen select period.

The first step in the construction of the select mortality table is the construction of the ultimate mortality table as discussed previously.

Thus, choose a starting age for the table, denoted by $\alpha$, and an arbitrary radix, denoted by $l_\alpha$.

As mentioned previously, the starting age will often be 0 (but this will depend very much on the nature of the population observed).

For example, the published version of the a(55) select mortality table begins at age 60 (although the full table contains data for ages 20 and upwards).

The reason for this is that the table is based on the mortality experience of annuitants, and individuals seldom take out annuity contracts prior to retirement (so that the majority of the population observed was aged 60 and upwards).

Then, for all ages $x \geq \alpha$, we calculate recursively the values of $l_x$ using $l_{x+1} = l_x \times (1 - q_x)$ and determine the values of $d_x$ using $d_x = l_x - l_{x+1}$.

When completed, this gives the ultimate portion of the table.

Suppose that the select period is $s$ years.

Using a deterministic interpretation of the life table, we use the $l_{[x]+r}$ (for $r < s$) to denote the number of lives who are alive at age $(x + r)$ from an initial group of $l_{[x]}$ lives selected at age $x$.

Then, we calculate the values of $l_{[x]}, l_{[x]+1}, \ldots, l_{[x]+(s-1)}$ recursively using:

- $$p_{[x]+(s-1)} = \frac{l_{[x]+s}}{l_{[x]+(s-1)}} \equiv \frac{l_{x+s}}{l_{[x]+(s-1)}} \Rightarrow l_{[x]+(s-1)} = \frac{l_{x+s}}{1 - q_{[x]+(s-1)}}$$

- $$p_{[x]+(s-2)} = \frac{l_{[x]+(s-1)}}{l_{[x]+(s-2)}} \Rightarrow l_{[x]+(s-2)} = \frac{l_{[x]+(s-1)}}{1 - q_{[x]+(s-2)}}$$

  $\vdots$

- $$p_{[x]} = \frac{l_{[x]+1}}{l_{[x]}} \Rightarrow l_{[x]} = \frac{l_{[x]+1}}{1 - q_{[x]}}$$

And, then we calculate $d_{[x]+r}$, for $r = 0,1,2,\ldots,(s-1)$, using $d_{[x]+r} = l_{[x]+r} - l_{[x]+r+1}$.

For example, another extract from the A1967-70 select life table is shown below.

| age $[x]$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{x+2}$ | age $x+2$ |
|---|---|---|---|---|
| 60 | 29615.936 | 29417.538 | 29132.138 | 62 |
| 61 | 29130.898 | 28920.265 | 28615.051 | 63 |
| 62 | 28600.975 | 28378.059 | 28052.632 | 64 |
| 63 | 28023.708 | 27788.571 | 27442.681 | 65 |
| 64 | 27396.808 | 27149.632 | 26783.206 | 66 |

We can easily recover the select and ultimate mortality rates considered above as follows:

(i) $$p_{[60]} = \frac{l_{[60]+1}}{l_{[60]}} \Rightarrow q_{[60]} = 1 - p_{[60]} = \frac{l_{[60]} - l_{[60]+1}}{l_{[60]}} = \frac{29615.936 - 29417.538}{29615.936} = 0.00669903$$

(ii) $$p_{[60]+1} = \frac{l_{62}}{l_{[60]+1}} \Rightarrow q_{[60]+1} = 1 - p_{[60]+1} = \frac{l_{[60]+1} - l_{62}}{l_{[60]+1}} = \frac{29417.538 - 29132.138}{29417.538} = 0.00970167$$

(iii) $$p_{62} = \frac{l_{63}}{l_{62}} \Rightarrow q_{62} = 1 - p_{62} = \frac{l_{62} - l_{63}}{l_{62}} = \frac{29132.138 - 28615.051}{29132.138} = 0.01774971$$

**1.15.3 Using select life table functions**

Previously, for mortality dependent on age only, we considered the use of the tabulated functions to calculate probabilities useful in life insurance mathematics. In particular, we have:

- ${}_n p_x = \frac{l_{x+n}}{l_x}$, represents the probability that a life of exact age $x$ will survive for at least $n$ years to reach exact age $(x+n)$; and

- ${}_{n|m}q_x = \dfrac{l_{x+n} - l_{x+n+m}}{l_x}$, represents the probability that a life of exact age $x$ will die in the $m$-year period between exact ages $(x+n)$ and $(x+n+m)$

A useful special case of the latter relationship is ${}_{n|}q_x = \dfrac{l_{x+n} - l_{x+n+1}}{l_x}$, which represents the probability that a life of exact age $x$ will die between exact ages $(x+n)$ and $(x+n+1)$.

Similar probabilities can be defined for a select mortality table, so that we have:

- ${}_n p_{[x]+r} = \dfrac{l_{[x]+r+n}}{l_{[x]+r}}$, represents the probability that a life of exact age $(x+r)$, that was a select life at exact age $x$, will survive for at least $n$ years to reach exact age $(x+r+n)$; and
  - if $r+n \geq s$ (where $s$ is the length of the select period), then we replace $l_{[x]+r+n}$ in the numerator by $l_{x+r+n}$
- ${}_{n|m}q_{[x]+r} = \dfrac{l_{[x]+r+n} - l_{[x]+r+n+m}}{l_{[x]+r}}$, represents the probability that a life of exact age $(x+r)$, that was a select life at exact age $x$, will die in the $m$-year period between exact ages $(x+r+n)$ and $(x+r+n+m)$
  - similar comment to that above applies if $r+n \geq s$ or $r+n+m \geq s$

**Example 1.15.1**

Using the A1967-70 mortality table, calculate ${}_{1|2}q_{[60]+1}$.

**Solution**

From above, ${}_{1|2}q_{[60]+1}$ represents the probability that a life of exact age 61, that was a select life at exact age 60, will die between exact ages 62 and 64.

Thus, we have ${}_{1|2}q_{[60]+1} = \dfrac{l_{[60]+1+1} - l_{[60]+1+1+2}}{l_{[60]+1}} = \dfrac{l_{[60]+2} - l_{[60]+4}}{l_{[60]+1}}$.

Now, as the A1967-70 table has a select period of years, we have $l_{[60]+2} = l_{62}$ and $l_{[60]+4} = l_{64}$.

Thus, we have ${}_{1|2}q_{[60]+1} = \dfrac{l_{62} - l_{64}}{l_{[60]+1}} = \dfrac{29132.138 - 28052.632}{29417.538} = 0.036696$.

□

## Exercises for Chapter 1

Exercise 1

The mortality in a certain life table is such that:

$$l_x = l_0 \times \left(1 - \frac{x}{110}\right)^{½}$$

(a) Determining the limiting age, ω

(b) Obtain an expression for $\mu_x$

(c ) Calculate $q_{70}$

Exercise 2

Weibull's law of mortality states that the force of mortality at age x, $\mu_x$, is given by:

$\mu_x = c\delta x^{\delta-1}$ for $x \geq 0$ where $c > 0$, $\delta > 1$.

(i) Given $\mu_{40} = 0.0025$ and $\mu_{60} = 0.02$, calculate the values of the parameters c and δ.

(ii) Hence, or otherwise, find the probability that a life of age 70 dies between ages 75 and 76.

Exercise 3

In a particular survival model, the force of mortality at age $x$, $\mu_x$, is assumed to be constant for all ages, $x$.

(i) Show that the complete expectation of life at age $x$, $\overset{o}{e}_x$, is constant for all ages, $x$.

(ii) Comment on whether or not you think that this is a suitable model for human mortality.

Exercise 4

A life table with a select period of 2 years is based on rates of mortality that satisfy the following relationship:

$$q_{[x-s]+s} = \frac{2}{4-s} \times q_x \qquad \text{for } s = 0,1$$

Suppose that $l_{68} = 100{,}000$.

Then, given $q_{65} = 0.025$, $q_{66} = 0.026$ and $q_{67} = 0.028$, calculate the following:

(i) $l_{67}$

(ii) $l_{[65]+1}$

(iii) $l_{[65]}$

## Solutions to exercises for Chapter 1

Exercise 1

(a) $\omega$ is the lowest age for which $l_x = 0$.

By inspection, $\omega$ = 110.

(b) From (1.8.2), $\mu_x = -\frac{1}{l_x} \cdot \frac{d}{dx}(l_x)$

With $l_x = l_0 \times \left(1 - \frac{x}{110}\right)^{1/2}$,

$$\mu_x = \frac{1}{2(110 - x)}$$

(c) $$q_{70} = 1 - \frac{l_{71}}{l_{70}} = 1 - \left( \frac{1 - \frac{71}{110}}{1 - \frac{70}{110}} \right)^{1/2}$$

$$= 1 - \left( \frac{39}{40} \right)^{1/2}$$

$$= 0.01258.$$

<u>Exercise 2</u>

(i) $$\mu_{40} = c\delta(40)^{\delta-1} = 0.0025$$

$$\mu_{60} = c\delta(60)^{\delta-1} = 0.02$$

$$\Rightarrow \frac{\mu_{60}}{\mu_{40}} = \frac{(60)^{\delta-1}}{(40)^{\delta-1}} = \frac{0.02}{0.0025} = 8$$

$$\Rightarrow \left( \frac{60}{40} \right)^{\delta-1} = 8 \Rightarrow (\delta - 1) \times l_n\left( \frac{60}{40} \right) = l_n(8)$$

$$\Rightarrow \delta - 1 = \frac{l_n(8)}{l_n\left( \frac{60}{40} \right)} \Rightarrow \delta = 6.128534.$$

$$\mu_{40} = 0.0025 = c\delta(40)^{\delta-1} \Rightarrow c = 2.4795 \times 10^{-12}.$$

(ii) $${}_{5|}q_{70} = \frac{l_{75} - l_{76}}{l_{70}} = {}_5p_{70} - {}_6p_{70}$$

$${}_tp_x = \exp\left( -\int_x^{x+t} \mu_s \, ds \right)$$

$$\Rightarrow {}_5p_{70} = \exp\left(-\int_{70}^{75} \mu_s \, d_s\right)$$

$$= \exp\left(-\int_{70}^{75} c\delta s^{\delta-1} \, ds\right)$$

$$= \exp\left(-c\left[s^{\delta}\right]_{s=70}^{s=75}\right)$$

$$= \exp\left[-c\left(75^{\delta} - 70^{\delta}\right)\right]$$

Similarly, ${}_6p_{70} = \exp\left[-c(76^{\delta} - 70^{\delta})\right]$

$c=2.4795\times10^{-12}$ and $\delta=6.128534$

$$\Rightarrow {}_5p_{70} = 0.767173$$

$$ {}_6p_{70} = 0.718894$$

$$\Rightarrow {}_{5|}q_{70} = {}_5p_{70} - {}_6p_{70} = 0.048279$$

Exercise 3

(i) We have $\overset{\circ}{e}_x = \int_0^{\infty} {}_t p_x dt$.

Now, we have ${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$.

As the force of mortality is constant for all ages, we have ${}_t p_x = \exp\left(-\int_0^t \mu ds\right) = e^{-\mu t}$.

Thus, we have $\overset{\circ}{e}_x = \int_0^{\infty} e^{-\mu t} dt = \left[-\frac{1}{\mu}e^{-\mu t}\right]_{t=0}^{t=\infty} = \frac{1}{\mu}$ (which is independent of age, $x$).

(ii) Clearly, this is not a reasonable model for human mortality. We would expect that the expectation of the complete future lifetime will reduce as age increases (or, equivalently, that the force of mortality will increase as age increases).

Exercise 4

(i) $$\frac{l_{68}}{l_{67}} = p_{67} = 1 - q_{67} = 0.972 \Rightarrow l_{67} = \frac{100{,}000}{0.972} = 102{,}881.$$

(ii) $$\frac{l_{67}}{l_{[65]+1}} = p_{[65]+1} = 1 - q_{[65]+1} = 1 - \frac{2}{4-1} \times q_{66} = 0.982667 \Rightarrow l_{[65]+1} = \frac{102{,}881}{0.982667} = 104{,}695.$$

(iii) $$\frac{l_{[65]+1}}{l_{[65]}} = p_{[65]} = 1 - q_{[65]} = 1 - \frac{2}{4-2} \times q_{65} = 0.9875 \Rightarrow l_{[65]} = \frac{104{,}695}{0.9875} = 106{,}021.$$