Dynamical Systems
Solutions to Exercises

1.

Figure 1: Phase diagrams for (i), (ii) and (iii) respectively. Only fixed point is at the origin since the equations are linear and homogeneous

2.

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<td>(iii)</td>
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<td>(v)</td>
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<td>(vi)</td>
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Table 1: For each equation the table indicates: linear; autonomous; dimension of phase space where applicable

3. (i)

\[
\int \frac{dy}{\cos^2 y} = \int x \, dx \quad \Rightarrow \quad \int \sec^2 y \, dy = \int x \, dx \quad \Rightarrow \quad \tan y = \frac{x^2}{2} + C
\]

\[x = 0, \quad y = \pi/4 \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad y = \tan^{-1}\left(\frac{x^2}{2} + 1\right)\]

The original equation is defined for all \(x\) and \(y\), thus the domain equals \(\mathbb{R}^2\).

\[\tan^{-1}\left(\frac{x^2}{2} + 1\right)\] defined for all \(x\) thus the interval of definition \(I = (-\infty, \infty)\).
(ii) Writing the equation in the form $\frac{dy}{dx} + \tan xy = \cos x$ we see that it is a first order linear equation with integrating factor given by

$$R = \exp \left( \int \tan x \, dx \right) = \exp (-\ln \cos x) = \frac{1}{\cos x}$$

Thus

$$\left( \frac{y}{\cos x} \right)' = 1 \Rightarrow \frac{y}{\cos x} = x + C \Rightarrow y = \cos x(x + C)$$

$x = 0$, $y = 1$ $\Rightarrow$ $C = 1 \Rightarrow y = (x + 1) \cos x$

The original equation is defined for all $y$, however due to $\tan x$, $x \neq (2k-1)\pi/2$.

Thus the domain is given by: $D = \{(x, y) \in \mathbb{R}^2 : x \neq (2k-1)\pi/2 \quad k \in \mathbb{Z}\}$

$(x + 1) \cos x$ is defined for all $x$ and must contain $x = 0$, since the condition is $x = 0$, $y = 1$.

Thus accounting for the restriction on $x$ in the domain the interval of definition $I = (-\pi/2, \pi/2)$.

(iii) $\int \frac{dy}{1 + y^2} = \int dx \Rightarrow \tan^{-1} y = x + C$

$x = 0$, $y = 1$ $\Rightarrow$ $C = \frac{\pi}{4} \Rightarrow y = \tan(x + \frac{\pi}{4})$

The original equation defined for all $x$ and $y$ thus domain equals $\mathbb{R}^2$

The interval of definition must contain zero thus the interval of definition is given by:

$$-\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4} \Rightarrow I = \left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$$

4. (i) Fixed points at $y \cos y = 0$ $\Rightarrow$ $y = 0$, or $y = (2k-1)\frac{\pi}{2}$ where $k$ is an integer.

To classify use either the change in sign of $y \cos y$ at each fixed point or consider the sign of $(y \cos y)'$ at each fixed point.

At $y = 0$, $y \cos y$ changes sign from negative to positive with increasing $y$: thus $y = 0$ is a repellor.

For the other fixed points I will use the derivative method, thus: $(y \cos y)' = \cos y - y \sin y$.

Bearing in mind that $\cos y = 0$ at each of these points:

At $\pm \frac{\pi}{2}, \pm \frac{5\pi}{2}, \pm \frac{9\pi}{2}, \ldots$, $(\cos y - y \sin y) < 0$, thus attractors

At $\pm \frac{3\pi}{2}, \pm \frac{7\pi}{2}, \pm \frac{11\pi}{2}, \ldots$, $(\cos y - y \sin y) > 0$, thus repellors

See Fig 2(i) for phase diagram.

(ii) Fixed points at $y^2(1 - y) = 0$ $\Rightarrow$ $y = 0$, or $y = 1$.

About $y = 0$, $y^2(1 - y)$ is non-negative with increasing $y$, thus $y = 0$ is an increasing shunt.

About $y = 1$, $y^2(1 - y)$ changes from positive to negative with increasing $y$, thus $y = 1$ is an attractor.

See Fig 2(ii) for phase diagram.
Figure 2: (i) Repellors at 0 and ±\(\frac{(4k-1)\pi}{2}\), attractors at ±\(\frac{(4k-3)\pi}{2}\), \(k = 1, 2, \ldots\) (ii) Shunt at 0, attractor at 1 (iii) diagram not defined at \(\frac{(2k-1)\pi}{2}\) \(k = \pm 1, \pm 2, \ldots\) All fixed points repellors (iv) attractor at −3 and repellor at +1.

(iii) Fixed points at \(\tan y = 0 \Rightarrow y = k\pi\) \(k \in \mathbb{Z}\). However since \(\tan y\) is not defined at \(\frac{(2k-1)\pi}{2}\) the phase diagram will have gaps at these points.

About each fixed point \(\tan y\) changes from negative to positive, thus each point is a repellor.
Alternatively using the derivative method \((\tan y)' = \sec^2 y > 0 \Rightarrow\) repellors.

See Fig 2(iii) for phase diagram.

(iv) Fixed points at \(y^2 + 2y - 3 = (y + 3)(y - 1) = 0 \Rightarrow y = -3\) \(y = 1\)

About \(y = -3\), \((y + 3)(y - 1)\) changes from positive to negative hence \(y = -3\) is an attractor.
About \(y = 1\), \((y + 3)(y - 1)\) changes from negative to positive hence \(y = 1\) is a repellor.

See Fig 2(iv) for phase diagram.

\[
\frac{dy}{dx} = y^2 + 2y - 3 = (y + 3)(y - 1) \Rightarrow \int \frac{dy}{(y + 3)(y - 1)} = \int dx
\]

Using partial fractions this gives:

\[
\left(\frac{-1}{4}\right)\int \frac{1}{y + 3} + \frac{1}{1 - y} \Rightarrow \left(\frac{-1}{4}\right) \ln \left(\frac{y + 3}{1 - y}\right) = x + C
\]

The initial condition \(x = 0, y = -1\) give \(C = 0\) hence:

\[
\ln \left(\frac{y + 3}{1 - y}\right) = -4x \Rightarrow \frac{y + 3}{1 - y} = e^{-4x} \Rightarrow y = \frac{e^{-4x} - 3}{1 + e^{-4x}}
\]

This is plotted as the exact solution in Fig 3 and lies between \(y = -3\) and \(y = 1\).
\[ y = -3 + 2e^{-4x} \]

**Figure 3:** Exact solution and fixed point solutions in heavy line. Note: linear solution about \( y = -3 \) close to exact solution about \( y = -3 \) but as \( x \) decrease and becomes negative the difference between the two solutions tends to infinity. Similarly for the linear solution about \( y = 1 \).

Since the denominator of the solution is never zero the interval of definition is \( I = (\infty, \infty) \).

**Linearisation**

In general, about a fixed point \( a \) of the system \( \frac{dy}{dx} = X(y) \), the linearisation is \( \frac{dz}{dx} = X'(a)z \), where \( y = a + z \).

In the example \( X(y) = y^2 + 2y - 3 \) \( \Rightarrow \) \( X'(y) = 2y + 2 \) thus the general linearisation about \( y = a \) is \( z' = 2(a + 1)z \).

The fixed points are given by \( X(a) = 0 \) \( \Rightarrow \) \( a = -3 \) \( a = 1 \) thus:

\[
\begin{align*}
(i) \quad & \Rightarrow z' = -4z \quad \text{about } z = -3 \quad (i) \quad \text{and} \quad z' = 4z \quad \text{about } z = 1 \quad (ii) \\
(ii) \quad & \Rightarrow z = Ae^{-4x} \quad \Rightarrow y = -3 + Ae^{-4x}. \quad x = 0, \quad y = -1 \quad \Rightarrow \quad A = 2 \quad \Rightarrow \quad y = -3 + 2e^{-4x} \\
& \Rightarrow z = Ae^{4x} \quad \Rightarrow y = 1 + Ae^{-4x}. \quad x = 0, \quad y = -1 \quad \Rightarrow \quad A = -2 \quad \Rightarrow \quad y = 1 - 2e^{4x}
\end{align*}
\]

These two linearisations are plotted in Fig3, where they are seen to be respectively close to the exact solution in the neighbourhood of \( y = -3 \) and \( y = 1 \).

6. (a) Eigenvalues given by the solution of

\[
\det \begin{pmatrix} 13 - \lambda & 9 \\ -18 & -14 - \lambda \end{pmatrix} = \lambda^2 + \lambda - 20 = 0
\]

Hence the eigenvalues are \( \lambda_1 = 4 \) and \( \lambda_2 = -5 \).
The eigenvector corresponding to $\lambda_1 = 4$ is given by:

\[
\begin{pmatrix}
13 - 4 & 9 \\
-18 & -14 - 4
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow
\begin{pmatrix}
9 & 9 \\
-18 & -18
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\Rightarrow
\]

\[
a = -b \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]

The eigenvector corresponding to $\lambda_2 = -5$ is given by:

\[
\begin{pmatrix}
13 + 5 & 9 \\
-18 & -14 + 5
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow
\begin{pmatrix}
18 & 9 \\
-18 & -9
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\Rightarrow
\]

\[
2a = -b \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -2a \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\]

Thus the matrix $P$ is given by $P = (E_1 E_2) = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$ and $J = \begin{pmatrix} 4 & 0 \\ 0 & -5 \end{pmatrix}$

(b) Eigenvalues given by the solution of

\[
\text{det} \begin{pmatrix} 5 - \lambda & 4 \\ -10 & -7 - \lambda \end{pmatrix} = \lambda^2 + 2\lambda + 5 = 0
\]

Hence the eigenvalues are $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$.

The eigenvector corresponding to $\lambda_1 = -1 + 2i$ is given by:

\[
\begin{pmatrix}
5 - (-1 + 2i) & 4 \\
-10 & -7 - (-1 + 2i)
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow
\begin{pmatrix}
6 - 2i & 4 \\
-10 & -6 - 2i
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\Rightarrow
\]

\[
(6 - 2i)a = -4b \Rightarrow \frac{a}{b} = \frac{4}{2i - 6} \quad \text{take} \quad E_1 = \begin{pmatrix} 4 \\ 2i - 6 \end{pmatrix} \Rightarrow E_2 = \begin{pmatrix} 4 \\ -2i - 6 \end{pmatrix}
\]

Taking the real and imaginary part of $E_1$ gives:

\[
e_1 = \text{Re}(E_1) = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \quad e_2 = \text{Im}(E_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}
\]

Thus the matrix $P$ is given by $P = (e_1 e_2) = \begin{pmatrix} 4 & 0 \\ -6 & 2 \end{pmatrix}$ and $J = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

(c) Eigenvalues given by the solution of

\[
\text{det} \begin{pmatrix} 3 - \lambda & 1 \\ -1 & 1 - \lambda \end{pmatrix} = (\lambda - 2)^2 = 0
\]

Hence a single eigenvalue $\lambda = 2$

The eigenvector corresponding to $\lambda = 2$ is given by:

\[
\begin{pmatrix}
3 - 2 & 1 \\
-1 & 1 - 2
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix}
\Rightarrow
\]

\[
-a = b \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{take} \quad E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

The Jordan vector $J_1$ is given by:

\[
\begin{pmatrix}
3 - 2 & 1 \\
-1 & 1 - 2
\end{pmatrix} J_1 = E \Rightarrow
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} =
\begin{pmatrix} 1 \\ -1 \end{pmatrix}
\Rightarrow
\]
\[ a + b = 1 \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 - b \\ b \end{pmatrix} = b \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{take } J_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

Thus the matrix \( P \) is given by

\[ P = (EJ_1) = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \quad \text{and } J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \]

7. (a) Writing \( Q(i) \) in the standard form and using \( Q6(a) \) we have:

\[ \dot{x} = \begin{pmatrix} 13 & 9 \\ -18 & -14 \end{pmatrix} x, \quad \lambda_1 = 4 \quad \lambda_2 = -5 \Rightarrow \text{saddle point} \]

\[ E_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow x = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + \beta \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t} \]

In Fig 4(a) the eigenlines corresponding to \( E_1 \) and \( E_2 \) are trajectories and since \( E_1 \) corresponds to the positive eigenvalue points move out from the origin along this line as indicated by the arrows.

(b) Writing \( Q(ii) \) in the standard form and using \( Q6(b) \) we have:

\[ \dot{x} = \begin{pmatrix} 5 & 4 \\ -10 & -7 \end{pmatrix} x, \quad \lambda_1 = -1 + 2i \quad \lambda_2 = -1 - 2i \Rightarrow \text{stable focus} \]

\[ \varepsilon_1 = \begin{pmatrix} 4 \\ -6 \end{pmatrix} \quad \varepsilon_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow x = r_0 e^{-t} \left\{ \cos(-2t + \phi_0) \begin{pmatrix} 4 \\ -6 \end{pmatrix} + \sin(-2t + \phi_0) \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \]

The trajectories spiral about the fixed point at the origin and are drawn into the origin as time increases. The orientation of the trajectories is not obvious at this point; it may spiral clockwise or anti-clockwise. Thus considering the trajectories that cut the positive \( x_2 \) axis, that is to say set \( x_1 = 0 \) and take \( x_2 > 0 \), and looking at the first equation \( \dot{x}_1 = 5x_1 + 4x_2 \) we see that \( \dot{x}_1 > 0 \). Thus \( x_1 \) is increasing and the trajectories move from left to right in this region. The arrows are entered onto the trajectories accordingly, Fig 4(b), to give a clockwise rotation.
(c) Writing $Q(i)$ in the standard form and using $Q6(a)$ we have:

$$\dot{x} = \left( \begin{array}{cc} 3 & 1 \\ -1 & 1 \end{array} \right) x, \quad \lambda = 2 \quad \Rightarrow \quad \text{unstable improper node}$$

$$E = \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \Rightarrow x = e^{2t}(at + b) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) + ae^{2t} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$$

Since there is only one eigenline the trajectories are asymptotically parallel to this line. In detail, since the node is unstable, as $t \rightarrow \infty$ the trajectories become parallel to $E$ and tend to infinity; as $t \rightarrow -\infty$ the trajectories enter the fixed point at the origin parallel to $E$. The vector $J_1$ plays no obvious part in constructing the diagram. Of more use are the points where the trajectories are horizontal and where they are vertical. These points occur where $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ respectively. From the equations we have therefore that the trajectories are horizontal when $-x_1 + x_2 = 0$ and vertical when $3x_1 + x_2 = 0$. These two straight lines are shown on the diagram Fig 4(c) as dotted lines.

The direction of the phase paths can be checked using the method of part(b), namely looking at the sign of $\dot{x}_1$ as the curves cut the positive $x_2$-axis.

8. (a)

$$\dot{x} = \left( \begin{array}{cc} -17 & 39 \\ -6 & 13 \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + \left( \begin{array}{c} 13 \\ 26 \end{array} \right)$$

$$a = -A^{-1} \left( \begin{array}{c} 13 \\ 26 \end{array} \right) = -\frac{1}{13} \left( \begin{array}{cc} 13 & -39 \\ 6 & -17 \end{array} \right) \left( \begin{array}{c} 13 \\ 26 \end{array} \right) = \left( \begin{array}{c} -13 \\ -6 \\ 17 \\ 2 \end{array} \right) = \left( \begin{array}{c} 65 \\ 28 \end{array} \right)$$

In general:

$$x = a + \bar{z} \quad \Rightarrow \quad \dot{x} = \dot{z} \quad \text{and} \quad A\dot{x} + \bar{b} = Aa + A\bar{z} + \bar{b} = -b + A\bar{z} + \bar{b} = A\bar{z} \quad \Rightarrow \quad \dot{z} = A\bar{z}$$

The Eigenvalues are given by the solution of $\det \left( \begin{array}{cc} -17 - \lambda & 39 \\ -6 & 13 - \lambda \end{array} \right) = \lambda^2 + 4\lambda + 13 = 0$

Hence the eigenvalues are $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$.

The eigenvector corresponding to $\lambda_1 = -2 + 3i$ is given by:

$$\left( \begin{array}{cc} -17 - (-2 + 3i) \\ -6 \\ 13 - (-2 + 3i) \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \Rightarrow \left( \begin{array}{cc} -15 - 3i \\ -6 \\ 15 - 3i \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

$$(15 + 3i)a = 39b \quad \Rightarrow \quad \frac{a}{b} = \frac{39}{15 + 3i} \quad \text{take} \quad E_1 = \left( \begin{array}{c} 39 \\ 15 + 3i \end{array} \right) \Rightarrow \quad E_2 = \left( \begin{array}{c} 39 \\ 15 - 3i \end{array} \right)$$

The fixed point is a stable focus since the eigenvalues are complex with negative real part.

Taking the real and imaginary part of $E_1$ gives:

$$e_1 = Re(E_1) = \left( \begin{array}{c} 39 \\ 15 \end{array} \right) \quad e_2 = Im(E_1) = \left( \begin{array}{c} 0 \\ -3 \end{array} \right) \Rightarrow$$

$$x = \bar{z} + a = r_0 e^{-2t} \left\{ \cos(-3t + \phi_0) \left( \begin{array}{c} 39 \\ 15 \end{array} \right) + \sin(-3t + \phi_0) \left( \begin{array}{c} 0 \\ -3 \end{array} \right) \right\} + \left( \begin{array}{c} 65 \\ 28 \end{array} \right)$$

In Fig 5(a) the orientation of the focus is obtained by considering the curves as they cut the positive $z_2$-axis (vertical dashed axis in fig). Thus with $z_1 = 0$ and $z_2 > 0$ and by considering $\dot{z}_1 = -17z_1 + 39z_2$ we deduce that $\dot{z}_1 > 0$. Thus as the curves cut the positive $z_2$-axis $z_1$ increases, hence the rotation is clockwise.
(b) \[
\dot{\mathbf{x}} = \begin{pmatrix} 0 & 6 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}
\]
\[
a = -A^{-1} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}
\]

Eigenvalues given by the solution of \[
\det \begin{pmatrix} 0 - \lambda & 6 \\ -1 & 5 - \lambda \end{pmatrix} = \lambda^2 - 5\lambda + 6 = 0
\]

Hence the eigenvalues are \(\lambda_1 = 3\) and \(\lambda_2 = 2\).

The eigenvector corresponding to \(\lambda_1 = 3\) is given by:
\[
\begin{pmatrix} -3 & 6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = 2b \quad \Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = b \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{take} \quad E_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

The eigenvector corresponding to \(\lambda_2 = 2\) is given by:
\[
\begin{pmatrix} -2 & 6 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = 3b \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = b \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{take} \quad E_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\]

Since the eigenvalues are real different and both positive the fixed point is an \textbf{unstable node}.

The solution is given by:
\[
\mathbf{x} = \bar{x} + a = a \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + b \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -4 \\ -1 \end{pmatrix}
\]

(c) \[
\dot{\mathbf{x}} = \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]
\[
a = -A^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -\begin{pmatrix} 2 & -9 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}
\]

Eigenvalues given by the solution of \[
\det \begin{pmatrix} -4 - \lambda & 9 \\ -1 & 2 - \lambda \end{pmatrix} = (\lambda + 1)^2 = 0
\]

Hence there is only a single eigenvalues \(\lambda = -1\).

The eigenvector corresponding to \(\lambda = -1\) is given by:
\[
\begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = 3b \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = b \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{take} \quad E = \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\]

Construct the \(J\) vector to form the new basis:
\[
(A - \lambda I)J = E \quad \Rightarrow \quad \begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\]
\[
\Rightarrow a = 3b - 1 \quad \Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3b - 1 \\ b \end{pmatrix} = b \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{take} \quad J = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]
Since the repeated eigenvalue is negative the fixed point is a \textbf{stable inflected (improper) node}. The solution is given by:

\[ x = z + a = (\alpha t + \beta) e^{-t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha e^{-t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} \]

9. For the linear system \( \dot{x} = Ax \) the origin is a fixed point. Since \( \det A = \cos^2 \theta + \sin^2 \theta = 1 \neq 0 \) the fixed point is simple.

\[ \det(A - \lambda I) = 0 \Rightarrow (\cos \theta - \lambda)^2 + \sin^2 \theta = 0 \Rightarrow \lambda - \cos \theta = \pm i \sin \theta \Rightarrow \lambda = \cos \theta \pm i \sin \theta \]

(a) star node \( \Rightarrow A = \lambda I \), thus \( \sin \theta = 0 \Rightarrow \theta = 0 \) or \( \pi \Rightarrow \lambda = \cos \theta \)

To be stable \( \cos \theta = \lambda < 0 \Rightarrow \theta = \pi \)

(b) As (a) but to be unstable \( \cos \theta = \lambda > 0 \Rightarrow \theta = 0 \)

(c) Centre if \( \lambda \) purely imaginary. Thus \( \cos \theta = 0 \Rightarrow \theta = \pi/2 \)

(d) Stable focus if \( \sin \theta \neq 0 \) and \( \cos \theta < 0 \). Thus \( \pi/2 < \theta < \pi \)

(e) Unstable focus if \( \sin \theta \neq 0 \) and \( \cos \theta > 0 \). Thus \( 0 < \theta < \pi/2 \)

10. In general for the system

\[ \dot{x}_1 = X_1(x_1, x_2) \quad \dot{x}_2 = X_2(x_1, x_2) \]

the Jacobian is given by

\[ J = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{pmatrix} \]

Thus:

(a) \[ J = \begin{pmatrix} 2x_1 x_2 + x_2 & x_1^2 + x_1 \\ \cos x_1 \cos x_2 & -\sin x_1 \sin x_2 \end{pmatrix} \]
(b) \( J = \begin{pmatrix} 2x_1x_2^2 + \sin x_2 - e^{x_1} & 2x_2x_1^2 + x_1 \cos x_2 \\ \sin x_2 - x_2 \sin x_1 & x_1 \cos x_2 + \cos x_1 \end{pmatrix} \)

(c) \( J = \begin{pmatrix} 1 + x_2 & 1 + x_1 \\ \sin x_2 - x_2 \sin x_1 & x_1 \cos x_2 + \cos x_1 \end{pmatrix} \)

11. In each case it is clear that \( X_1(0, 0) = 0 \) and \( X_2(0, 0) = 0 \), thus the origin is a fixed points for each of the systems in Q9. Evaluating each of the above \( J \) at the origin gives the three linearisations as follows:

(a) \( \dot{z} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z \). Since \( \det J = 0 \) the fixed point is not simple, hence no classification.

(b) \( \dot{z} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} z \). Since \( \det J = -1 \neq 0 \) the fixed point is simple. The eigenvalues of \( J \) are \( \pm 1 \) thus the origin is a saddle point.

(c) \( \dot{z} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z \). Since \( \det J = 1 \neq 0 \) the fixed point is simple. \( J \) is already in the canonical form for an unstable improper node. (single eigenvalue = 1)

12. The Eigenvalues are given by the solution of \( \det \begin{pmatrix} 1 - \lambda & 3 \\ -6 & -5 - \lambda \end{pmatrix} = \lambda^2 + 4\lambda + 13 = 0 \)

Hence the eigenvalues are \( \lambda_1 = -2 + 3i \) and \( \lambda_2 = -2 - 3i \).

The eigenvector corresponding to \( \lambda_1 = -2 + 3i \) is given by:

\[ \begin{pmatrix} 1 - (-2 + 3i) \\ -6 \end{pmatrix} \begin{pmatrix} 3 \\ -5 - (-2 + 3i) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 - 3i \\ -6 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\((3 - 3i)a = -3b \Rightarrow \frac{a}{b} = \frac{-3}{3 - 3i} = \frac{-1}{1 - i} \)

Take \( E_1 = \begin{pmatrix} -1 \\ 1 - i \end{pmatrix} \Rightarrow E_2 = \begin{pmatrix} -1 \\ 1 + i \end{pmatrix} \)

Thus

\[ P = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} \]

\[ \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = -2y_1 + 3y_2 \quad \text{(i)} \]

Substitute \( y_1 = r \cos \theta \) and \( y_2 = r \sin \theta \) into (i) and (ii) to give:

\[ \dot{r} \cos \theta - r \dot{\theta} \sin \theta = -2r \cos \theta + 3r \sin \theta \quad \text{(iii)} \]

\[ \dot{r} \sin \theta + r \dot{\theta} \cos \theta = -3r \cos \theta - 2r \sin \theta \quad \text{(iv)} \]

\[ \cos \theta \times \text{(iii)} + \sin \theta \times \text{(iv)} \Rightarrow \dot{r} = -2r \Rightarrow r = r_0 e^{-2t} \]
\[
\sin \theta \times (iii) - \cos \theta \times (iv) \Rightarrow -r \dot{\theta} = 3r \Rightarrow \dot{\theta} = -3 \Rightarrow \theta = -3t + \theta_0
\]

Thus: 
\[
y_1 = r_0 e^{-2t} \cos(-3t + \theta_0) \quad \text{and} \quad y_2 = r_0 e^{-2t} \sin(-3t + \theta_0)
\]

The solution to \( \dot{x} = Ax \) is given by:
\[
x(t) = r_0 e^{-2t} \left\{ \cos(-3t + \theta_0) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \sin(-3t + \theta_0) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}
\]

where \( r_0 \) and \( \theta_0 \) are arbitrary constants.

13. Fixed points at \( x_1 \cos x_2 = 0 \) and \( x_2 \cos x_1 = 0 \). In the given region this give five fixed points:

\[
(0,0) \quad \left( \pm \frac{\pi}{2}, \pm \frac{\pi}{2} \right) \quad \left( -\frac{\pi}{2}, \pm \frac{\pi}{2} \right) \quad \left( -\frac{\pi}{2}, -\frac{\pi}{2} \right)
\]

The Jacobian is given by
\[
J = \begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ -x_2 \sin x_1 & \cos x_1 \end{pmatrix}
\]

Thus the linearisation about the fixed points are:

\[
(0,0) \quad \dot{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z \\
(\pm \pi, \pm \pi) \quad \dot{z} = \begin{pmatrix} 0 & -\frac{\pi}{2} \\ -\frac{\pi}{2} & 0 \end{pmatrix} z \\
(\pm \frac{\pi}{2}, \mp \frac{\pi}{2}) \quad \dot{z} = \begin{pmatrix} 0 & \frac{\pi}{2} \\ \frac{\pi}{2} & 0 \end{pmatrix} z
\]

Clearly \((0,0)\) is an unstable star node.

At \( (\pm \frac{\pi}{2}, \pm \frac{\pi}{2}) \) the eigenvalues are given by 
\[
\det \begin{pmatrix} -\lambda & -\frac{\pi}{2} \\ -\frac{\pi}{2} & -\lambda \end{pmatrix} = \lambda^2 - \left( \frac{\pi}{2} \right)^2 = 0 \Rightarrow \lambda = \pm \frac{\pi}{2} \text{ hence saddle points}
\]

At \( (\pm \frac{\pi}{2}, \mp \frac{\pi}{2}) \) the eigenvalues are given by 
\[
\det \begin{pmatrix} -\lambda & \frac{\pi}{2} \\ \frac{\pi}{2} & -\lambda \end{pmatrix} = \lambda^2 - \left( \frac{\pi}{2} \right)^2 = 0 \Rightarrow \lambda = \pm \frac{\pi}{2} \text{ hence saddle points}
\]

From the original equations
\[
\frac{dx_2}{dx_1} = \frac{x_2 \cos x_1}{x_1 \cos x_2}
\]

Clearly \( x_2 = x_1 \) satisfies this equation with each side equal to +1. Similarly \( x_2 = -x_1 \) satisfies the equation with each side equal to \(-1\).

The \( x_1 \)-axis is given by \( x_2 = 0 \). Substituting this into the original equations gives \( \dot{x}_1 = x_1 \) for the first equation and trivially for the second equation \( 0 = 0 \). \( \dot{x}_1 = x_1 \Rightarrow x_1 = Ce^t \) thus if \( C > 0 \) this generates the positive \( x_1 \)-axis and if \( C < 0 \) the negative \( x_1 \)-axis. Thus the \( x_1 \)-axis is a trajectory.

Similarly the \( x_2 \)-axis is trajectory.

We have established that both the axes and the lines \( x_2 = \pm x_1 \) are trajectories. These along with the fixed points are exact trajectories and can be added with certainty onto the phase diagram. The details from the linearisations are then used to construct a reasonable approximation to the global
phase diagram. In some cases it is not always clear how the linear diagrams link together to give a global diagram. Remember the linear phase diagrams are only approximations to the true picture and only remain good near their respective fixed points.

The saddles at each of the four fixed points can be positioned using the eigenvectors of the linearised system about each point. The usual calculation leads to the eigenvectors at the two points \((\pm \frac{\pi}{2}, \pm \frac{\pi}{2})\) being \(E_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\) and \(E_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) relative to the axes through the points. As we see in each case the eigenvector \(E_2\) lies along the trajectory of the non-linear system given by \(x_2 = x_1\). However the eigenvector \(E_1\) is not a trajectory of the non-linear system about either fixed point and can only be used as a guide when drawing the saddle. These are shown in Fig 6 as dotted lines.

Similar remarks are also applicable at the other two fixed point, namely \((\pm \frac{\pi}{2}, \mp \frac{\pi}{2})\).

The completed diagram is shown in Fig 6. The arrows can be put on the diagram fairly easily by noting for example that the origin is unstable and all trajectories leave this point with increasing \(t\).

![Figure 6](image)

**Figure 6**: Q13. The \(x_1\) and \(x_2\) axes are trajectories; the two lines \(x_2 = \pm x_1\) are trajectories; Saddles at the four fixed points \((\pm \frac{\pi}{2}, \frac{\pi}{2})\) and \((\pm \frac{\pi}{2}, -\frac{\pi}{2})\); star node at the origin

14. Fixed points at \(x_1(1-x_1x_2) = 0 = x_2(1-x_1x_2)\). For all points on the hyperbola \(x_2 = 1/x_1, x_1x_2-1 = 0\) thus all points are fixed points.

\[
J = \begin{pmatrix}
1 - 2x_1x_2 & -x_1^2 \\
-x_2^2 & 1 - 2x_1x_2
\end{pmatrix} \implies \det J_{(0,0)} = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0 \text{ thus the origin is simple}
\]

\[
\det J = 1 + 3x_1^2x_2^2 - 4x_1x_2 \quad \text{on the hyperbola} \quad x_1x_2 = 1 \implies \det J = 1 + 3 - 4 = 0
\]
Thus all points on the hyperbola are non-simple fixed points.

Setting $x_2 = 0$ in Eq(1): The second equation is satisfied and the first implies $\dot{x}_1 = x_1 \Rightarrow x_1 = Ae^t$, which gives the whole of the $x_1$ axis for different values of $A$.

Similarly the $x_2$ axis is a trajectory.

From Eq(1)

$$\frac{dx_2}{dx_1} = \frac{x_2(1 - x_1 x_2)}{x_1(1 - x_1 x_2)} = \frac{x_2}{x_1} \quad 1 - x_1 x_2 \neq 0$$

Solving:

$$\int \frac{dx_2}{x_2} = \int \frac{dx_1}{x_1} \quad x_1, x_2 \neq 0 \Rightarrow x_2 = Ax_1$$

This gives all straight lines through the origin, omitting the origin and points on the hyperbola.

Figure 7: Note the apparent change of direction of the trajectories as they approach the hyperbola of fixed points.

15. No solution given.