MA3606 DIFFERENTIAL EQUATIONS

BASED LARGELY ON NOTES BY ALAN BRYAN

Contents

Many important problems in engineering, the physical sciences, economics and financial mathematics are expressed in terms of differential equations. For example:

- **Newton's second law in dynamics:**

  $$m\frac{d^2x}{dt^2} = F\left(t, x(t), \frac{dx}{dt}\right),$$

  where $x(t)$ is position, t is time, F is force.
- **Schrödinger's equation in quantum mechanics:**

  $$\frac{d^2\psi}{dx^2} + (E - V(x))\psi = 0,$$

  where $\psi(x)$ is wave function, $V(x)$ is potential energy, E is energy.
- **Black-Scholes equation in financial mathematics:**

  $$\frac{\partial V}{\partial t} + x^2\frac{\partial^2 V}{\partial x^2} + x\frac{\partial V}{\partial x} - V = 0,$$

  where x is price of stock, t is time, $V(x,t)$ is price of option.
- **Korteweg-deVries equation in fluid dynamics:**

  $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

  where x is spacial coordinate, t is time, $u(x,t)$ is profile of wave. More generally, the **Navier-Stokes equations** are a coupled system of nonlinear partial differential equations describing the velocity of a fluid in **fluid dynamics**.

The first two examples are **ordinary differential equations (ODEs)**, and the last two are **partial differential equations (PDEs)**.

In this course, we shall be mainly concerned with linear differential equations, the mathematical theory of which is highly developed. Nonlinear differential equations, particularly nonlinear PDEs, are usually very difficult to deal with.

Part I: ORDINARY DIFFERENTIAL EQUATIONS

1. **ODEs**: FIRST ORDER EQUATIONS

Consider equations of the form

$$y'(x) = f(x, y(x)), \tag{1.1}$$

where f is a specified function of x and y.

Any differentiable function $y = \phi(x)$ that satisfies equation 1.1 for all x in some interval (possibly infinite) is called a **solution**. Its graph is a curve in (x, y)-space.

Firstly, we shall consider the case when $f(x, y)$ is a **linear function of** y, *i.e.*,

$$f(x, y) = -p(x)y + g(x).$$

1.1. **Linear equations. Exact solution.**

Equation (1.1) may be rewritten as

$$y'(x) + p(x)y(x) = g(x) \tag{1.2}$$

multiply this equation by the function $\mu(x)$:

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)g(x),$$

and define $\mu(x)$ so that

$$\text{LHS} = \frac{d}{dx}(\mu(x)y(x)) = \mu(x)y'(x) + \mu'(x)y(x).$$

This requires

$$\mu'(x) = p(x)\mu(x).$$

Hence

$$\mu(x) = \exp\left\{\int p(x)\,dx\right\}. \tag{1.3}$$

With this multiplier, equation (1.2) becomes

$$\frac{d}{dx}(\mu(x)y(x)) = \mu(x)g(x).$$

Integrating gives

$$\mu(x)y(x) = \int \mu(x)g(x)\,dx + C,$$

where C is a constant. So

$$y(x) = \frac{\int \mu(x)g(x)\,dx + C}{\mu(x)} \tag{1.4}$$

is the solution to (1.2). This is the **integrating factor method**.

Example 1.1.

$$y' + 2y = e^{-x}.$$

Solution:

$$y = e^{-x} + Ce^{-2x}.$$

An **initial condition** picks out a particular curve. Generally, a first-order equation (1.1) with an initial condition, *i.e.*,

$$y' = f(x, y(x)), \qquad y(x_0) = \eta_0 \tag{1.5}$$

is called an **initial-value problem**. The solution curve $y = \phi(x, x_0)$ passes through the point (x_0, η_0).

Example 1.2.

$$y' + 2y = e^{-x}, \qquad y(0) = \frac{3}{4}.$$

Solution:

$$y = e^{-x} - \frac{1}{4}e^{-2x}.$$

Equation (1.4) shows how to construct solutions to a linear DE. We may then determine the constant C to satisfy an initial condition.

Alternatively, the IVP for the linear DE can be solved as follows: as before equation (1.2) becomes, by using the integrating factor,

$$\frac{d}{dx}\left(\mu(x)y(x)\right) = \mu(x)g(x).$$

Integrating from x_0 to x gives

$$\mu(x)y(x) - \mu(x_0)y(x_0) = \int_{x_0}^{x} \mu(s)g(s)\,ds,$$

giving the solution

$$y(x) = \frac{1}{\mu(x)}\left\{\mu(x_0)y(x_0) + \int_{x_0}^{x} \mu(s)g(s)\,ds\right\}.$$

We have proved the following theorem

Theorem 1.3. *The solution of the IVP*

$$y'(x) + p(x)y(x) = g(x), \qquad y(x_0) = \eta_0$$

is given by

$$y(x) = \frac{1}{\overline{\mu}(x)}\left\{\eta_0 + \int_{x_0}^{x} \mu(s)g(s)\,ds\right\},$$

where $\overline{\mu}(x) = \exp\left\{\int_{x_0}^{x} p(s)\,ds\right\}$.

Next we consider some general questions about the IVP (1.5) for a general function $f(x, y)$ (*i.e.*, not necessarily linear):

- does the IVP have a solution? (**EXISTENCE**)
- is there more than one solution? (**UNIQUENESS**)
- is the solution valid for all x or only for some interval about the initial point?

These questions are addressed in the theorem in the next section.

1.2. **The existence and uniqueness theorem: Picard iteration.** Consider the IVP

$$y' = f(x, y(x)), \qquad y(x_0) = \eta_0. \tag{1.6}$$

In general we will not be able to explicitly determine the solution of (1.6), particularly if the ODE is nonlinear. In such cases we need another means of determining the existence, or not, of a solution.

The following iterative procedure leads to an existence theorem and also gives approximations to the solution. It has three stages:

(1) construct a sequence of functions $y_n(x)$ which get closer and closer to the solution;
(2) show that $y_n(x)$ has a limit $y(x)$ on a suitable interval $x_0 \le x \le x_0 + \alpha$;
(3) show that $y(x)$ is the solution of (1.6) on this interval.

We shall consider only stage (1).

We need to rewrite (1.6) as

$$y(x) = L[x, y(x)] \tag{1.7}$$

and look for the **fixed points** of (1.7) (rather like the method of solving functional equations by rearrangement and iteration, e.g. Newton-Raphson method). That is, we define a sequence $y_n(x)$ by

$$y_{n+1}(x) = L[x, y_n(x)]. \tag{1.8}$$

Then if $y_n(x) \to y(x)$ as $n \to \infty$, $y(x)$ solves (1.6) via (1.7).

So what is $L[x, y_n(x)]$? Integrating (1.6) over $[x_0, x]$ we have

$$y(x) = \int_{x_0}^{x} f(s, y(s))\, ds + \eta_0. \tag{1.9}$$

Coversely, if $y(x)$ satisfies (1.9) and is continuous then $y(x)$ satisfies (1.6).

Equation (1.9) is an integral equation and is in the form (1.7) if we set

$$L[x, y(x)] = \int_{x_0}^{x} f(s, y(s))\, ds + \eta_0. \tag{1.10}$$

We therefore define a sequence of function $y_1(x), y_2(x), \ldots$ by

$$y_{n+1}(x) = \eta_0 + \int_{x_0}^{x} f(s, y_n(s))\, ds. \tag{1.11}$$

The initial function $y_0(x)$ is usually taken to be the constant function $y_0(x) = \eta_0$. The functions $y_n(x)$ are called the **Picard iterates** after the French mathematician who discovered them. It may be proved that the sequence of Picard iterates, with certain conditions on $f(x, y(x))$, has a limit, and that this limiting function satisfies the IVP (1.6). This proves **existence**.

Finally, we must prove **uniqueness**. Suppose that (1.6) has more than one solution, say $y(x)$ and $z(x)$. Then

$$y(x) = \int_{x_0}^{x} f(s, y(s))\, ds + \eta_0, \qquad z(x) = \int_{x_0}^{x} f(s, z(s))\, ds + \eta_0.$$

Subtracting these equations we have

$$\begin{aligned}|y(x)-z(x)| &= \left|\int_{x_0}^{x} f(s,y(s))-f(s,z(s))\,ds\right| \\ &\leq \int_{x_0}^{x} |f(s,y(s))-f(s,z(s))|\,ds \\ &= \int_{x_0}^{x} \left|\frac{\partial f}{\partial y}(s,\rho(s))\,(y(s)-z(s))\right|\,ds \\ &\leq K\int_{x_0}^{x} |(y(s)-z(s))|\,ds,\end{aligned}$$

where $\rho(s)\in(y(s),z(s))$ and $K=\max\left|\frac{\partial f}{\partial y}\right|$ for (x,y) in some rectangle R. Let

$$w(x)=|y(x)-z(x)| \quad \text{and} \quad W(x)=\int_{x_0}^{x} w(s)\,ds \tag{1.12}$$

Then the above inequality can be rewritten as

$$w(x)\leq K\int_{x_0}^{x} w(s)\,ds$$

or

$$\frac{dW}{dx}\leq KW(x).$$

Multiplying both sides by $e^{-K(x-x_0)}$ gives

$$\frac{d}{dx}\left(e^{-K(x-x_0)}W(x)\right)\leq 0,$$

and integrating from x_0 to x gives

$$e^{-K(x-x_0)}W(x)-W(x_0)\leq 0. \tag{1.13}$$

But from (1.12), $W(x_0)=0$ and $W(x)$ is nonnegative. Hence, from (1.13), $W(x)=0$. This means that $w(x)=|y(x)-z(x)|=0$ for each x, and so $y(x)=z(x)$. This proves uniqueness.

The three stages of proof on the Picard iterates, together with the above proof of uniqueness establish the following theorem.

Theorem 1.4 (Existence and uniqueness (Picard))**.** *Suppose* $f(x,y)$ *and* $\frac{\partial f}{\partial y}$ *are continuous in the rectangle*

$$R=\{(x,y): x_0\leq x\leq x_0+a, |y-\eta_0|\leq b\}.$$

Let $M=\max_{(x,y)\in R}|f(x,y)|$ *and set* $\alpha=\min\left(a,\frac{b}{M}\right)$. *Then the initial-value problem*

$$y'=f(x,y(x)), \qquad y(x_0)=\eta_0$$

has a unique solution on the interval $x_0\leq x\leq x_0+\alpha$.

Example 1.5.

$$y'=y^{\frac{1}{2}}, \qquad y(0)=0.$$

Example 1.6.

$$y' = y^2, \qquad y(0) = 1.$$

Example 1.7.

$$\frac{dy}{dx} = \frac{\sin(xy)}{4+x^2}, \qquad y(2) = 1.$$

1.3. **Linear equations - existence and uniqueness.**

Recall that the IVP is

$$y'(x) + p(x)y(x) = g(x), \qquad y(x_0) = \eta_0. \tag{1.14}$$

In this case

$$f(x,y) = -p(x)y + g(x),$$
$$\frac{\partial f}{\partial y} = -p(x).$$

Thus, f and $\frac{\partial f}{\partial y}$ are both continuous provided $p(x)$ and $g(x)$ are continuous, and we have the following theorem.

Theorem 1.8. *Let $p(x)$ and $g(x)$ be continuous on an interval I containing x_0. Then the IVP (1.14) has a unique solution throughout I.*

Example 1.9.

$$y' + 2y = e^{-x}, \qquad y(0) = \frac{3}{4}.$$

Example 1.10.

$$y' + \frac{1}{x}y = e^x, \qquad y(1) = 2.$$

2. SECOND ORDER EQUATIONS

A second order ODE is an equation of the form

$$\frac{d^2y(x)}{dx^2} = f\left(x, y(x), \frac{dy(x)}{dx}\right).$$

We shall only consider linear equations. In this case f is a linear function in y and y', *i.e.*,

$$f(x,y,y') = -p(x)y'(x) - q(x)y(x) + g(x),$$

so that the general form is

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x), \tag{2.1}$$

with initial conditions of the form

$$y(x_0) = \eta_0, \qquad y'(x_0) = \eta_0'. \tag{2.2}$$

Continuing from the previous section, we begin with an existence and uniqueness theorem.

Theorem 2.1. *Let $p(x)$, $q(x)$ and $g(x)$ be continuous on an interval $I \subseteq \mathbb{R}$. Then the initial value problem (2.1) and (2.2) has exactly one solution, and this exists throughout I.*

Example 2.2.

$$y''(x) + \frac{1}{x-3}y'(x) + \sqrt{x}y(x) = \ln(x+1), \qquad y(1) = 3, \quad y'(1) = -5.$$

2.1. **Homogeneous linear equations.** In this case $g(x) \equiv 0$ in (2.1), so that the equation is

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

The problem of solving the homogeneous equation is the fundamental one since once this has been solved it is always possible to solve the corresponding non-homogeneous problem (at least in terms of an integral).

The easiest type of problem to solve is one in which the equation has constant coefficients and we consider that first.

2.1.1. *Constant coefficients.*

Example 2.3.

$$2y'' + 3y' + y = 0.$$

General solution:

$$y(x) = Ae^{-x} + Be^{-\frac{1}{2}x}.$$

Generally, the roots of the auxiliary equation determine the nature of the solution:

(1) Real roots, $m_1 \neq m_2$: $\quad y(x) = Ae^{m_1 x} + Be^{m_2 x}$.
(2) Real roots, $m_1 = m_2$: $\quad y(x) = e^{m_1 x}(A + Bx)$.
(3) Complex roots, $m = \alpha \pm i\beta$: $\quad y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

Example 2.4. (1) $y'' + k^2 y = 0 \implies y(x) = A\cos kx + B\sin kx$.
(2) $y'' - k^2 y = 0 \implies y(x) = A\cosh kx + B\sinh kx$.

2.1.2. *Fundamental solutions.* We introduce the following operator notation:

Let $p(x)$ and $q(x)$ be continuous functions on an open interval $I \subseteq \mathbb{R}$. Then, for any function $\phi(x)$ which is twice differentiable on I we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{2.3}$$

L may be thought of as an operator which maps one function to another. Using this notation, we shall discuss differential equations of the form

$$L[y] = y'' + py' + qy = 0 \tag{2.4}$$

with initial conditions

$$y(x_0) = \eta_0, \qquad y'(x_0) = \eta_0'. \tag{2.5}$$

Theorem 2.5 (Principle of superposition). *If y_1 and y_2 are solutions of (2.4), then so is the linear combination $c_1y_1 + c_2y_2$, for any values of the constants c_1 and c_2.*

Proof.

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ &= c_1L[y_1] + c_2L[y_2] \\ &= 0, \end{aligned}$$

since $L[y_1] = 0 = L[y_2]$. ☐

Now, is it possible to choose the constants c_1 and c_2 to satisfy (2.5)? These conditions require

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = \eta_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = \eta_0' \end{cases} \tag{2.6}$$

These equations can be solved for c_1 and c_2 **provided that the coefficient matrix is nonsingular**. Thus, with c_1 and c_2 given by solving (2.6), the linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies the initial value problem (2.4) and (2.5) provided that

$$y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0) \neq 0.$$

Definition 2.6. *The determinant*

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x) y_2'(x) - y_1'(x) y_2(x)$$

is called the ***Wronskian*** *of y_1 and y_2.*

We have used the concept of 'general' solution to an ODE, e.g. $2y'' + 3y' + y = 0$ has general solution $y = Ae^{-x} + Be^{-\frac{1}{2}x}$. How do we know these are all the solutions?

Theorem 2.7 (General solution)**.** *If y_1 and y_2 are two solutions of the linear DE (2.4) and if there is a point x_0 at which $W(y_1, y_2)(x_0) \neq 0$ then the family of solutions $y(x) = c_1 y_1(x) + c_2 y_2(x)$ with arbitrary coefficients c_1 and c_2 includes every solution of (2.4).*

Proof. Let $\phi(x)$ be any solution of (2.4). Then, since $W(y_1, y_2)(x_0) \neq 0$, there exist c_1, c_2 such that

$$\begin{aligned} y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) &= \phi(x_0), \\ y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) &= \phi'(x_0). \end{aligned}$$

Since $\phi(x)$ and $c_1 y_1(x) + c_2 y_2(x)$ are two solutions satisfying the same initial conditions, then, by uniqueness (Theorem 2.1), they must be equal. ☐

Thus to find a general solution of any second order homogeneous linear ODE we need only find two solutions whose Wronskian is nonzero at some point.

Definition 2.8. *Two solutions y_1 and y_2 of (2.4) with $W(y_1, y_2)(x_0) \neq 0$, for some $x_0 \in I$, form a* ***fundamental set of solutions (FSS)****.*

Example 2.9.

$$2y'' + 3y' + y = 0; \qquad y_1 = e^{-x}, \quad y_2 = e^{-\frac{1}{2}x}.$$

Example 2.10.

$$2x^2 y'' + 3xy' - y = 0; \qquad y_1 = \frac{1}{x}, \quad y_2 = x^{\frac{1}{2}}.$$

2.1.3. *Linear independence and the Wronskian.* Two functions f and g are **linearly dependent** on an interval I if there exists constants k_1 and k_2 (not both zero) such that

$$k_1 f(x) + k_2 g(x) = 0, \quad x \in I.$$

The functions f and g are **linearly independent** if they are not linearly dependent.

Theorem 2.11. *If f and g are differentiable on the interval I and $W(f,g)(x_0) \neq 0$ for some $x_0 \in I$, then f and g are linearly independent on I.*

Proof. Suppose that f and g are linearly dependent. Then

$$\begin{aligned} k_1 f(x) + k_2 g(x) &= 0, \\ k_1 f'(x) + k_2 g'(x) &= 0, \end{aligned}$$

for some nonzero k_1, k_2, the first equation by definition and the second by differentiating the first. The coefficient matrix must be singular, *i.e.*$f(x)g'(x) - f'(x)g(x) = 0$, *i.e.*$W(f,g)(x) = 0$, for all $x \in I$. This contradicts the asssumption that $W(f,g)(x_0) \neq 0$ for some $x_0 \in I$. □

It follows from the theorem that the functions y_1, y_2 in a fundamental set of solutions are linearly independent. But what about the converse? That is, if two solutions are linearly independent do they form a fundamental set?

Theorem 2.11 is not true in general as the following counterexample shows: Let

$$f(x) = \begin{cases} x^2, & x < 0, \\ 0, & x \geq 0, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & x < 0, \\ x^2, & x^2 \geq 0. \end{cases}$$

Clearly $f(x) \neq kg(x)$ so f, g are linearly independent. But $W(f,g)(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix} = 0$ for $x < 0$, and $W = \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix}$ for $x \geq 0$, so that $W(f,g)(x) = 0$ for all $x \in \mathbb{R}$. Hence linear independence of f, g does not imply $W(f,g)(x_0) \neq 0$ for some x_0.

However the converse **is** true if the functions are **solutions of a linear second order differential equation**. Indeed, suppose y_1 and y_2 are solutions of (2.4) such that $W(y_1, y_2)(x_0) = 0$. Then

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{cases}$$

for some nonzero c_1, c_2. Thus both of the functions $c_1 y_1 + c_2 y_2$ and 0 are solutions of the IVP (2.4) and (2.5), with $\eta_0 = \eta_0' = 0$. Hence by uniqueness (Theorem 2.1), $c_1 y_1 + c_2 y_2 = 0$, i.e., y_1 and y_2 are linearly dependent.

We have proved the following equivalence.

Theorem 2.12. *Let y_1, y_2 be solutions of (2.4). Then y_1, y_2 form a fundamental set of solutions if and only if y_1, y_2 are linearly independent.*

Finally we have the following result on the Wronskian:

Theorem 2.13 (Abel)**.** *Let y_1, y_2 be two solutions of the DE (2.4):*

$$y'' + py' + qy = 0.$$

Then

$$W(y_1, y_2)(x) = c \exp\left\{ -\int_{x_0}^{x} p(s)\, ds \right\},$$

where $c = W(y_1, y_2)(x_0)$.

So the Wronskian of the functions in **any** fundamental set of solutions is determined (up to a constant multiple) by $p(x)$.

Proof. Using the definition of the Wronskian

$$\begin{aligned}\frac{dW}{dx} &= \frac{d}{dx}\left(y_1 y_2' - y_1' y_2\right)\\ &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'\\ &= y_1\left(-p y_2' - q y_2\right) - y_2\left(-p y_1' - q y_1\right)\\ &= -p\left(y_1 y_2' - y_1' y_2\right)\\ &= -pW.\end{aligned}$$

Hence

$$\frac{1}{W}\frac{dW}{dx} = -p(x).$$

Integrating from x_0 to x gives

$$\ln W(x) - \ln W(x_0) = -\int_{x_0}^{x} p(s)\, ds.$$

Exponentiating gives

$$W(x) = W(x_0) \exp\left\{ -\int_{x_0}^{x} p(s)\, ds \right\}.$$

□

It follows from the theorem that if y_1, y_2 are solutions then either $W(y_1, y_2) \equiv 0$ for all $x \in I$, or $W(y_1, y_2)(x)$ is never zero on I.

To recap: if y_1 and y_2 are solutions of the DE (2.4), then following four statements are equivalent:

- $c_1 y_1 + c_2 y_2$ is a general solution of (2.4)
- $W(y_1, y_2)(x_0) \neq 0$ for some $x_0 \in I$
- $W(y_1, y_2)(x) \neq 0$ for all $x \in I$
- y_1 and y_2 are linearly independent

Theorem 2.13 also gives an explicit recipe to produce a FSS from a single solution:

Theorem 2.14. *If* y_1 *is a nonzero solution of (2.4), then a second (linearly independent) solution is given by*

$$y_1 \int \left\{ y_1^{-2} \exp\left(-\int p(x)\, dx \right) \right\} dx.$$

Proof. Writing

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2},$$

with, from Abel's theorem,

$$W(x) = a\exp\left(-\int p\,dx\right),$$

then

$$\frac{y_2}{y_1} = \int \frac{W(x)}{y_1^2}\,dx + c.$$

Hence

$$\begin{aligned} y_2 &= y_1\left\{\int \frac{W(x)}{y_1^2}\,dx\right\} + cy_1 \\ &= ay_1\int\left\{y_1^{-2}\exp\left(-\int p(x)\,dx\right)\right\}dx + cy_1 \end{aligned}$$

□

2.2. **Nonhomogeneous linear equations.** We now consider the nonhomogeneous second order linear ODE

$$L[y] = y''(x) + p(x)y'(x) + q(x)y(x) = g(x), \tag{2.7}$$

where $p(x)$, $q(x)$ and $g(x)$ are given continuous functions an open interval $I \subseteq \mathbb{R}$.

Theorem 2.15. *The general solution of (2.7) can be written in the form*

$$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x) \tag{2.8}$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation $L[y] = 0$, c_1 and c_2 are arbitrary constants and y_p is a specific solution of the nonhomogeneous equation (2.7).

Proof. Let $\phi(x)$ be any solution of (2.7). Since

$$L[\phi - y_p] = L[\phi] - L[y_p] = g(x) - g(x) = 0$$

then $\phi - y_p$ is a solution of the corresponding homogeneous equation. Hence, by theorem 2.7,

$$\phi(x) - y_p(x) = c_1y_1(x) + c_2y_2(x)$$

for suitable constants c_1 and c_2. □

The specific solution $y_p(x)$ is usually called the **Particular Integral**. How do we determine $y_p(x)$? We shall consider two methods, the first for constant coefficient equations and the second more general.

2.2.1. *Method of undetermined coefficients.* This basically amounts to 'guessing' the form of the particular integral, and is the method covered in First Year Calculus.

Example 2.16.

$$2y'' + 3y' + y = x(1 + x)$$

Generally the form of $g(x)$ determines the choice of $y_p(x)$:

$g(x)$	$y_p(x)$
x^n	$Ax^n + \dots$
e^{mx}	$Ax^s e^{mx}$
$\sin\beta x,\ \cos\beta x$	$x^s(A\cos\beta x + B\sin\beta x)$

Here s is the smallest integer that ensures that no term in $y_p(x)$ is a solution of the homogeneous equation.

Example 2.17.

$$2y'' + 3y' + y = e^{-x}$$

2.2.2. *Method of variation of parameters.* This is a more general approach which also enables us to solve a wider class of constant coefficient equations.

Consider again the non-homogeneous second order linear ODE:

$$L[y] = y''(x) + p(x)y'(x) + q(x)y(x) = g(x). \tag{2.9}$$

Suppose we know the general solution of the associated homogeneous equation $L[y] = 0$, say

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x). \tag{2.10}$$

Note that it is **not** usually easy to find for variable coefficient equations.

Now replace the constants c_1 and c_2 by functions $u_1(x)$ and $u_2(x)$ and choose these so that

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{2.11}$$

satisfies (2.9). Differentiating (2.11) gives

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'. \tag{2.12}$$

Now we wish to substitute (2.11) into (2.9) in order to determine u_1 and u_2, but we have two unknowns and only one equation. We therefore choose a condition on u_1 and u_2, letting

$$u_1'y_1 + u_2'y_2 = 0. \tag{2.13}$$

Then

$$y' = u_1y_1' + u_2y_2'$$

and

$$y'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'.$$

Substituting into (2.9) and collecting terms gives

$$u_1\left(y_1'' + py_1' + qy_1\right) + u_2\left(y_2'' + py_2' + qy_2\right) + u_1'y_1' + u_2'y_2' = g(x)$$

and therefore

$$u_1'y_1' + u_2'y_2' = g(x). \tag{2.14}$$

Now (2.13) and (2.14) are a pair of simultaneous equations in u_1' and u_2'. Solving these we obtain

$$u_1' = -\frac{y_2(x)g(x)}{W(y_1, y_2)(x)}, \qquad u_2' = \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}.$$

Integration then gives $u_1(x)$ and $u_2(x)$. Finally substituting these expressioins back into (2.11) gives the particular integral. We have proved:

Theorem 2.18 (Variation of parameters)**.** *If y_1 and y_2 are fundamental solutions of the equation $L[y] = 0$ then the equation*

$$L[y] = y'' + py' + qy = g(x)$$

has the particular integral

$$y_p(x) = -y_1(x) \int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}\, dx + y_2(x) \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}\, dx \tag{2.15}$$

and general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

.

Example 2.19.

$$y'' + y = \tan x, \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

2.2.3. *Initial Value Problems.*

Theorem 2.20 (Initial value problem)**.** *Consider the initial value problem*

$$L[y] = y'' + py' + qy = g(x), \qquad \alpha < x < \beta, \tag{2.16}$$

$$y(x_0) = 0, \quad y'(x_0) = 0. \tag{2.17}$$

If y_1 and y_2 are fundamental solutions of the homogeneous equation $L[y] = 0$ then the solution of the IVP (2.16), (2.17) is

$$y(x) = \int_{x_0}^{x} \left\{ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(y_1, y_2)(t)} \right\} g(t)\, dt \tag{2.18}$$

Proof. The particular integral obtained in Theorem 2.18 has the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

with

$$y'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x).$$

The initial conditions (2.17) require

$$y(x_0) = u_1(x_0)y_1(x_0) + u_2(x_0)y_2(x_0) = 0$$
$$y'(x_0) = u_1(x_0)y_1'(x_0) + u_2(x_0)y_2'(x_0) = 0$$

and are only satisfied if $u_1(x_0) = 0$ and $u_2(x_0) = 0$, since the existence of a FSS implies that the determinant $W(y_1, y_2)(x_0)$ of the coefficient matrix of the system of equations is nonzero. Therefore

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt, \quad u_2(x) = \int_{x_0}^{x} \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}\, dt,$$

and

$$\begin{aligned} y(x) &= -y_1(x) \int_{x_0}^{x} \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt + y_2(x) \int_{x_0}^{x} \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\, dt \\ &= \int_{x_0}^{x} \left\{ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(y_1, y_2)(t)} \right\} g(t)\, dt. \end{aligned}$$

□

Corollary 2.21. *If the initial conditions (2.17) are replaced by*

$$y(x_0) = \eta_0, \quad y'(x_0) = \eta_0'. \tag{2.19}$$

then the solution of the IVP (2.16), (2.19) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

where $y_p(x)$ *is given by (2.18) in 2.20 and the constants* c_1 *and* c_2 *satisfy*

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= \eta_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= \eta_0' \end{aligned}$$

Example 2.22.

$$y'' + \omega^2 y = g(x), \qquad y(0) = y'(0) = 0.$$

Example 2.23.

$$y'' + \omega^2 y = g(x), \qquad y(0) = 2, \quad y'(0) = -1.$$

A summary of topics covered in this section:

- existence and uniqueness theorem for linear equations
- homogeneous equations
  - Wronskian; Abel's theorem
  - fundamental set of solutions $\{y_1, y_2\}$
  - general solution $y = c_1 y_1 + c_2 y_2$
- nonhomogeneous equations
  - general solution $y = c_1 y_1 + c_2 y_2 + y_p$
  - variation of parameters for $y_p(x)$
  - IVP

3. Higher order linear equations

Most of the results we have discussed for second order equations carry over into higher orders. We consider the ODE

$$L_n[y] = \frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = g(x). \tag{3.1}$$

with initial conditions

$$y(x_0) = \eta_0, \quad y'(x_0) = \eta_0', \quad \ldots, \quad y^{(n-1)}(x_0) = \eta_0^{(n-1)}. \tag{3.2}$$

Theorem 3.1 (Existence and Uniqueness)**.** *If* $p_1, \ldots, p_n, g$ *are continuous on* $I \subseteq \mathbf{R}$ *then the IVP (3.1) and (3.2) has a unique solution which exists throughout* I*.*

Example 3.2.

$$x(x-2)y''' - 3xy'' + 6x^2 y' - (\cos x)y = \sqrt{x+5},$$

$$y(1) = 1, \quad y'(1) = 0, \quad y''(1) = -1.$$

3.1. **Homogeneous equations.** Consider the homogeneous equation

$$L_n[y] = 0. \tag{3.3}$$

Definition 3.3. *The determinant*

$$W(y_1,\dots,y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is the **Wronskian** *of* $y_1,\dots,y_n$.

Definition 3.4. *Solutions* $y_1,\dots,y_n$ *of* $L_n[y]=0$ *with* $W(y_1,\dots,y_n)(x_0) \neq 0$ *for some* $x_0 \in I$ *form a* **fundamental set of solutions***.*

Theorem 3.5 (General solution)**.** *If* $\{y_1,\dots,y_n\}$ *is a FSS then every solution has the form*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \dots + c_ny_n(x),$$

where $c_1,\dots,c_n$ *are constants.*

Example 3.6.

$$y''' + y'' + y' + y = 0.$$

Theorem 3.7 (Abel's identity)**.** *If* $y_1,\dots,y_n$ *are solutions of* $L_n[y]=0$*, then*

$$W(y_1,\dots,y_n)(x) = c\exp\left\{-\int p_1(x)\,dx\right\},$$

where c *is a constant.*

Example 3.8. For the DE $y''' + y'' + y' + y = 0$ in the previous example, $p_1(x) = 1$, so $\int p_1(x)\,dx = \int 1\,dx = x$, giving $W(x) = ce^{-x}$.

3.2. **Nonhomogeneous equations:** $L_n[y] = g$**.**

Theorem 3.9 (General solution)**.** *If* $\{y_1,\dots,y_n\}$ *is a FSS of* $L_n[y]=0$ *then every solution of* $L_n[y]=g$ *has the form*

$$y(x) = c_1y_1(x) + \dots + c_ny_n(x) + y_p(x),$$

where $y_p(x)$ *is a specific solution of* $L_n[y]=g$ *and* $c_1,\dots,c_n$ *are constants.*

Theorem 3.10 (IVP)**.** *If* $\{y_1,\dots,y_n\}$ *is a FSS of* $L_n[y]=0$ *then the solution of the IVP*

$$L_n[y] = g(x), \qquad y(x_0) = y'(x_0) = \dots = y^{(n-1)}(x_0) = 0$$

is

$$y(x) = \int_{x_0}^{x} \frac{\widetilde{W}(y_1,\dots,y_n)(t,x)}{W(y_1,\dots,y_n)(t)}\,g(t)\,dt$$

where $W(y_1,\dots,y_n)(t)$ *is the Wronskian of* $y_1(t),\dots,y_n(t)$ *and*

$$\widetilde{W}(y_1,\dots,y_n)(t,x) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix}$$

is the Wronskian with the last row replaced by $y_1(x), \ldots, y_n(x)$*.*

4. INITIAL VALUE PROBLEMS: THE INFLUENCE FUNCTION

We shall consider second order DEs

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = g(x), \quad \alpha < x < \beta, \tag{4.1}$$

where $a_0(x), a_1(x), a_2(x)$ are continuous and $a_2(x) \neq 0$. Every such equation can be rewritten in **self-adjoint form** and these have useful general properties.

4.1. Self-adjoint differential equations.

Definition 4.1. *A second order DE is* **self-adjoint** *if it has the form*

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = h(x). \tag{4.2}$$

Equation (4.1) can be expressed in self-adjoint form (4.2) by multiplying it by an integrating factor, as follows:

Divide by a_2 to obtain

$$y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = \frac{g}{a_2}.$$

Then multiply by $p = \exp\left\{\int \frac{a_1}{a_2}\,dx\right\}$ to give

$$\frac{d}{dx}\left[p\frac{dy}{dx}\right] + p\frac{a_0}{a_2}y = p\frac{g}{a_2}.$$

Example 4.2. (1) $y'' + y = 0$.

(2) $x^2y'' + xy' + y = e^x, \quad x > 0$.

(3) $(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \quad -1 < x < 1$.

4.2. Homogeneous equations.

$$[p(x)y'(x)]' + q(x)y(x) = 0 \tag{4.3}$$

Theorem 4.3 (Wronskian)**.** *The Wronskian of the self-adjoint DE (4.3) is given by*

$$W(y_1, y_2)(x) = \frac{c}{p(x)},$$

where c is a constant.

Proof. Writing (4.3) as

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = 0,$$

Abel's formula (Theorem 2.13) gives

$$W(x) = c\exp\left\{-\int\frac{p'(x)}{p(x)}\,dx\right\} = c\exp\{-\ln p(x)\} = \frac{c}{p(x)}.$$

□

Theorem 4.4 (Second solution)**.** *If* y_1 *is a solution of (4.3) then a second solution is given by*

$$y_2(x) = y_1(x) \int \frac{dx}{p(x)[y_1(x)]^2}.$$

Proof. For any two solutions y_1 and y_2,

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W(y_1, y_2)(x)}{y_1^2} = \frac{c}{p y_1^2}.$$

Integration gives

$$\frac{y_2}{y_1} = \int \frac{c}{p y_1^2}\, dx$$

and put $c = 1$. ☐

4.3. **Nonhomogeneous equations.**

Consider the IVP

$$[p(x)y'(x)]' + q(x)y(x) = h(x) \tag{4.4}$$

$$y(x_0) = y'(x_0) = 0 \tag{4.5}$$

Writing (4.4) as

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = \frac{h}{p},$$

then, using Theorem 2.20, the solution of the IVP (4.4) and (4.5) is given by

$$y(x) = \int_{x_0}^{x} \left\{ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(y_1, y_2)(t)} \right\} \frac{h(t)}{p(t)}\, dt.$$

Thus, we can re-express Theorem 2.20 for the solution of the self-adjoint IVP as follows:

Theorem 4.5 (Influence function)**.** *The solution of the self-adjoint IVP*

$$[p(x)y'(x)]' + q(x)y(x) = h(x), \quad y(x_0) = y'(x_0) = 0$$

is given by

$$y(x) = \int_{x_0}^{x} R(x,t)h(t)\, dt \tag{4.6}$$

where the **influence function** $R(x,t)$ *is defined by*

$$R(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{p(t)W(y_1, y_2)(t)} \tag{4.7}$$

Example 4.6. (See example (2.23))

$$y'' + y = f(x), \qquad y(0) = y'(0) = 0.$$

Example 4.7.

$$x^2y'' + xy' - y = f(x), \quad x > 0, \qquad y(2) = y'(2) = 0.$$

Alternatively, the influence function can be characterised by the following properties:

(1) $R(x,t)$ satisfies the associated homogeneous DE with respect to x

(2) $R(x,t)|_{x=t} = 0$

(3) $\frac{\partial R}{\partial x}|_{x=t} = \frac{1}{p(t)}$

Indeed, writing (4.7) as

$$R(x,t) = -\frac{y_2(t)}{p(t)W(t)}y_1(x) + \frac{y_1(t)}{p(t)W(t)}y_2(x)$$

the three properties are clearly satisfied by the influence function.

Conversely, if a function $R(x,t)$ satisfies (1), (2) and (3), then

$$\begin{aligned} (1) \quad &\implies \quad R(x,t) = c_1(t)y_1(x) + c_2(t)y_2(x) \\ (2) \quad &\implies \quad R(t,t) = c_1(t)y_1(t) + c_2(t)y_2(t) = 0 \\ (3) \quad &\implies \quad \frac{\partial R}{\partial x}|_{x=t} = c_1(t)y_1'(t) + c_2(t)y_2'(t) = \frac{1}{p(t)} \end{aligned}$$

Solving simultaneously gives

$$c_1 = \frac{-y_2(t)}{p(t)W(t)}, \quad c_2 = \frac{y_1(t)}{p(t)W(t)},$$

as required.

Example 4.8. Reconsider Example 4.6:

$$y'' + y = f(x), \qquad y(0) = y'(0) = 0.$$

5. Boundary value problems

The problem of finding a solution to a second order linear DE

$$y''(x) + p_1(x)y'(x) + p_2(x)y(x) = g(x), \quad \alpha < x < \beta \tag{5.1}$$

that satisfies the boundary conditions

$$\begin{cases} a_{11}y(\alpha) + a_{12}y'(\alpha) + b_{11}y(\beta) + b_{12}y'(\beta) = c_1 \\ a_{21}y(\alpha) + a_{22}y'(\alpha) + b_{21}y(\beta) + b_{22}y'(\beta) = c_2 \end{cases} \tag{5.2}$$

is called **two-point boundary value problem**.

Here $I = (\alpha, \beta)$ and the conditions (5.2) are the most general we could consider. (Typically most of the constants a_{ij}, b_{ij} are zero.) When $c_1 = c_2 = 0$ the boundary conditions are said to be **homogeneous**.

Four special types of conditions occur frequently in applications:

- **Dirichlet:**
$$y(\alpha) = c_1, \quad y(\beta) = c_2$$
- **Neumann:**
$$y'(\alpha) = c_1, \quad y'(\beta) = c_2$$
- **Robin (or mixed):**
$$a_1y(\alpha) + a_2y'(\alpha) = c_1, \quad b_1y(\beta) + b_2y'(\beta) = c_2$$
- **Periodic:**
$$y(0) = y(2T), \quad y'(0) = y'(2T),$$
usually in some type of 'circular' geometry, where the period is $2T$.

For IVPs the Existence and Uniqueness Theorem 2.1 ensures a unique solution on some interval. For BVPs this is not necessarily the case.

For the homogeneous BVP (i.e. $g(x) \equiv 0$, $c_1 = c_2 = 0$ in (5.1), (5.2)) three cases can arise:

(1) a unique solution
(2) a one-parameter family of solutions
(3) a two-parameter family of solutions

For the nonhomogeneous BVP there is, in addition, a fourth possibility

(4) there are no solutions

For the homogeneous BVP, case (1) means that the only solution is the **trivial solution** $y(x) \equiv 0$.

Example 5.1. (1) $y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y(\pi/4) = 1.$
(2) $y'' + y = \cos 2x, \quad y'(0) = 0, \quad y'(\pi) = 0.$
(3) $y'' + 4y = 0, \quad y(0) = y(\pi), \quad y'(0) = y'(\pi).$
(4) $y'' + 4y = 4x, \quad y(0) = y(\pi), \quad y'(0) = y'(\pi).$

6. Eigenvalue problems

A type of homogeneous BVP of particular interest, since they arise in the solution of PDEs, is a problem containing a parameter, say λ, which will have a problem for some values of λ.

Definition 6.1. *Suppose the DE (5.1) contains a parameter* λ. *Then the values of* λ *for which the BVP (5.1), (5.2) has a non-trivial solution are called the* **eigenvalues** *and the corresponding solutions are the* **eigenfunctions**.

Example 6.2.

$$y'' + \lambda y = 0, \quad 0 < x < L, \quad \lambda \in \mathbf{R}$$

$$y(0) = y(L) = 0$$

Example 6.3.

$$y'' + \lambda y = 0, \quad 0 < x < L$$

$$y'(0) = y'(L) = 0$$

Example 6.4.

$$y'' + \lambda y = 0, \quad -\pi < x < \pi$$

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$$

The next example illustrates that even for a simple ODE, the eigenvalues may be difficult to determine exactly.

Example 6.5.

$$y'' + \lambda y = 0, \quad 0 < x < 1$$

$$y(0) = 0, \quad 3y(1) - y'(1) = 0$$

7. NONHOMOGENEOUS BOUNDARY VALUE PROBLEMS: THE GREEN'S FUNCTION

Consider the BVP

$$Ly := \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x) = -h(x), \quad \alpha < x < \beta \tag{7.1}$$

$$\begin{cases} B_1 y := a_1 y(\alpha) + a_2 y'(\alpha) = \eta_1 \\ B_2 y := b_1 y(\beta) + b_2 y'(\beta) = \eta_2 \end{cases} \tag{7.2}$$

(The RHS of (7.1) is written $-h(x)$. Not necessary, but conventional.)

7.1. Existence and uniqueness.

The general solution of (7.1) is given by Theorem 2.15 as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) - \int_\alpha^x R(x,t)h(t)\,dt \tag{7.3}$$

where y_1, y_2 form a FSS of the homogeneous equation and the third term is the particular integral obtained by the method of variation of parameters with $R(x,t)$ the influence function.

The boundary conditions (7.2) are satisfied if

$$\begin{cases} B_1 y = & c_1 B_1 y_1 + c_2 B_1 y_2 = \eta_1 \\ B_2 y = & c_1 B_2 y_1 + c_2 B_2 y_2 - b_1 \int_\alpha^\beta R(\beta,t)h(t)\,dt - b_2 \left\{ \int_\alpha^\beta \frac{\partial R}{\partial x} h(t)\,dt + R(\beta,\beta)h(\beta) \right\} = \eta_2 \end{cases} \tag{7.4}$$

These equations have a unique solution for c_1 and c_2 provided that

$$\Delta := \begin{vmatrix} B_1 y_1 & B_1 y_2 \\ B_2 y_1 & B_2 y_2 \end{vmatrix} \neq 0 \tag{7.5}$$

Thus, if $\Delta \neq 0$ then the BVP (7.1) and (7.2) has a unique solution given by (7.3).

Now let us consider what happens if $\Delta = 0$. If $\Delta = 0$ then the equations

$$\begin{aligned} c_1 B_1 y_1 + c_2 B_1 y_2 &= 0 \\ c_1 B_2 y_1 + c_2 B_2 y_2 &= 0 \end{aligned}$$

have solutions other than $c_1 = c_2 = 0$. The corresponding function $v(x) = \tilde{c}_1 y_1(x) + \tilde{c}_2 y_2(x)$ satisfies the **homogeneous** BVP $Lv = 0, \quad B_1 v = B_2 v = 0$. This means that if $y(x)$ is a solution of the nonhomogeneous BVP (7.1) and (7.2) then $y(x) + cv(x)$ is **also** a solution for any constant c. So the problem does not have a unique solution.

We have proved the following theorem:

Theorem 7.1. *The BVP (7.1), (7.2) has a unique solution if and only if the determinant*

$$\Delta = \begin{vmatrix} a_1 y_1(\alpha) + a_2 y_1'(\alpha) & a_1 y_2(\alpha) + a_2 y_2'(\alpha) \\ b_1 y_1(\beta) + b_2 y_1'(\beta) & b_1 y_2(\beta) + b_2 y_2'(\beta) \end{vmatrix} \neq 0$$

where y_1 and y_2 form a FSS of the associated homogeneous equation.

Note: if $\Delta = 0$ there may be no solution or infinitely many solutions.

In the case of a unique solution, the solution itself is given by equation (7.3) with c_1 and c_2 determined by equations (7.4). This can either be used as a procedure to solve the BVP or taken through to obtain a formula for the solution. However both of these methods are algebraically rather messy.

An alternative approach is to define an intermediary function, the Green's function, and to express the solution in terms of it. This may be shown to be equivalent to using equations (7.3) and (7.4) – we shall not give the proof.

7.2. **Green's function.** Consider the BVP (7.1): $Ly = -h(x)$ with homogeneous BCs

$$\begin{cases} a_1 y(\alpha) + a_2 y'(\alpha) = 0 \\ b_1 y(\beta) + b_2 y'(\beta) = 0 \end{cases} \tag{7.6}$$

Definition 7.2. *The* **Green's function** $G(x,t)$ *of the BVP is defined by the following conditions:*

(1) $G(x,t)$ satisfies $Ly = 0$ with respect to x $(x \neq t)$
(2) $G(x,t)$ satisfies the boundary conditions at $x = \alpha$ and $x = \beta$
(3) $G(x,t)$ is continuous at $x = t$
(4) $\frac{\partial G}{\partial x}$ is discontinuous at $x = t$ with $\frac{\partial G}{\partial x}|_{x=t+0} - \frac{\partial G}{\partial x}|_{x=t-0} = -\frac{1}{p(t)}$.

Then the solution of the BVP is given by the following theorem:

Theorem 7.3. *The solution of the BVP (7.1), (7.6) is given by*

$$y(x) = \int_\alpha^\beta G(x,t)h(t)\,dt \tag{7.7}$$

Example 7.4.

$$y'' = -f(x), \quad 0 < x < 1, \qquad y(0) = y(1) = 0.$$

Example 7.5.

$$y'' + n^2 y = -f(x), \quad 0 < x < 1, \quad n \in \mathbf{N}, \qquad y(0) = y(1) = 0.$$

Example 7.6. Solve Example (7.5) in the particular case when $f(x) = 1$.

$$\begin{aligned} y(x) &= \int_0^1 G(x,t)f(t)\,dt \\ &= \int_0^x \frac{\sin nt \sin n(1-x)}{n \sin n} \cdot 1\,dt + \int x1 \frac{\sin n(1-t)\sin nx}{n \sin n} \cdot 1\,dt \\ &= \frac{\sin n(1-x)}{n \sin n} \int_0^x \sin nt\,dt + \frac{\sin nx}{n \sin n} \int_x^1 \sin n(1-t)\,dt \\ &= \frac{\sin n(1-x)}{n \sin n} \left[\frac{-\cos nx + 1}{n} \right] + \frac{\sin nx}{n \sin n} \left[\frac{1 - \cos n(1-x)}{n} \right] \\ &= -\frac{1}{n^2} + \frac{\sin n(1-x) + \sin nx}{n^2 \sin(n)} \\ &= -\frac{1}{n^2} + \frac{\cos \frac{1}{2}n(1-2x)}{n^2 \cos \frac{1}{2}n} \\ &= \frac{2 \sin \frac{1}{2}n(1-x) \sin \frac{1}{2}x}{n^2 \cos \frac{1}{2}n}. \end{aligned}$$

We now highlight a symmetry property of the Green's function. Let

$$G(x,t) = \begin{cases} G_\alpha(x,t), & \alpha \le x \le t \\ G_\beta(x,t), & t \le x \le \beta \end{cases}$$

Then

$$G_\alpha(t,x) = G_\beta(x,t) \tag{7.8}$$

In Example 7.4

$$G(x,t) = \begin{cases} (1-t)x, & 0 \le x \le t \\ t(1-x), & t \le x \le 1 \end{cases}$$

In Example 7.5

$$G(x,t) = \begin{cases} \frac{\sin n(1-t)\sin nx}{n\sin n}, & 0 \le x \le t \\ \frac{\sin nt \sin n(1-x)}{n \sin n}, & t \le x \le 1 \end{cases}$$

This property may be used, to advantage, in the construction of a Green's function.

Example 7.7.

$$y'' - y = -f(x), \quad 0 < x < L, \qquad y'(0) = y(L) = 0.$$

We return now to the original BVP (7.1), (7.2) in which the boundary conditions are non-homogeneous.

Theorem 7.8. *The solution of the BVP (7.1), (7.2) is given by*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_\alpha^\beta G(x,t)h(t)\,dt \tag{7.9}$$

where y_1 and y_2 form a FSS of the corresponding homogeneous equation, $G(x,t)$ is the Green's function using the corresponding homogeneous boundary conditions, and the values of c_1, c_2 are chosen to satisfy the BCs.

Proof. Let $y_h(x) = c_1y_1(x) + c_2y_2(x)$ and $y_p(x) = \int_\alpha^\beta G(x,t)h(t)\,dt$. Then

$$\begin{aligned} Ly &= Ly_h + Ly_p = 0 + (-h) \\ B_1y &= B_1y_h + B_1y_p = B_1y_h + 0 = \eta_1 \\ B_2y &= B_2y_h + B_2y_p = B_2y_h + 0 = \eta_2 \end{aligned}$$

□

Example 7.9.

$$y'' - y = -f(x), \quad 0 < x < L, \qquad y'(0) = 1, \quad y(L) = 2.$$

Finally, we complete this section by proving Theorem 7.3: We write (7.1) as

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = -\frac{h}{p} \tag{7.10}$$

Using the method of variation of parameters, we proved in Theorem 2.18 that a particular integral of (7.10) is given by

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1'(x) = \frac{y_2(x)h(x)}{p(x)W(y_1,y_2)(x)} \qquad u_2'(x) = -\frac{y_1(x)h(x)}{p(x)W(y_1,y_2)(x)}$$

where y_1 and y_2 are fundamental solutions of the homogeneous problem and we have put $g(x) = -\frac{h(x)}{p(x)}$. Using Theorem 4.3 we can put $p(x)W(y_1,y_2)(x) = c$, where c is a constant.

Since $u_1(x), u_2(x)$ are only determined up to a constant then a particular integral of (7.10) is given by

$$\begin{aligned} y(x) &= y_1(x)\left\{-\int_x^\beta \frac{y_2(t)h(t)}{c}\,dt\right\} + y_2(x)\left\{\int_\alpha^x \frac{y_1(t)h(t)}{c}\,dt\right\} \\ &= \int_\alpha^x \left\{\frac{-y_1(t)y_2(x)}{c}\right\} h(t)\,dt + \int_x^\beta \left\{\frac{-y_1(x)y_2(t)}{c}\right\} h(t)\,dt \end{aligned}$$

Sicne $\Delta \neq 0$, we may choose the FSS y_1, y_2 so that

$$\begin{cases} a_1y_1(\alpha) + a_2y_1'(\alpha) = 0 \\ b_1y_2(\beta) + b_2y_2'(\beta) = 0 \end{cases} \tag{7.11}$$

and with this choice $y(x)$ satisfies the BCs (7.6).

So the solution may be written

$$y(x) = \int_\alpha^\beta G(x,t)h(t)\,dt$$

where

$$G(x,t) = \begin{cases} -y_1(x)y_2(t)/c, & \alpha \le x \le t \\ -y_1(t)y_2(x)/c, & t \le x \le \beta \end{cases} \tag{7.12}$$

Clearly $G(x,t)$ satisfies conditions (1),(2),(3) of our definition of a Green's function. Condition (4) is also satisfied since

$$\begin{aligned} \frac{\partial G}{\partial x}|_{x=t+0} - \frac{\partial G}{\partial x}|_{x=t-0} &= \frac{-y_1(t)y_2'(x)}{c}|_{x=t+0} - \frac{-y_1'(x)y_2(t)}{c}|_{x=t-0} \\ &= -\frac{[y_1(t)y_2'(t) - y_1'(t)y_2(t)]}{c} \\ &= -\frac{W(y_1,y_2)(t)}{c} \\ &= -\frac{1}{p(t)}. \end{aligned}$$

We also see from (7.12) that $G(x,t)$ satisfies the symmetry condition (7.8).

8. Sturm-Liouville problems

The problems we considered in §6 are special cases of **Sturm-Liouville** boundary value problems. These are eigenvalue problems of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x) + \lambda r(x)y(x) = 0, \quad \alpha < x < \beta \tag{8.1}$$

$$\begin{cases} a_1 y(\alpha) + a_2 y'(\alpha) = 0 \\ b_1 y(\beta) + b_2 y'(\beta) = 0 \end{cases} \tag{8.2}$$

where $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are continuous on $\alpha \le x \le \beta$.

The problem is **regular** if $p(x) > 0$ and $r(x) > 0$ for $\alpha \le x \le \beta$. The problem is **singular** if $p(x) > 0$ and $r(x) > 0$ for $\alpha < x < \beta$ and $r(\alpha) = 0$ or $r(\beta) = 0$.

We shall only consider regular problems. General properties are discussed in the following sections.

8.1. **Eigenfunctions and eigenvalues.**

We begin by proving two important properties.

Definition 8.1. *A set of functions* $\{f_n(x)\}_{n=1}^{\infty}$ *is said to be* **orthogonal** *with respect to the non-negative weight function* $w(x)$ *on the interval* $\alpha \le x \le \beta$ *if*

$$\int_{\alpha}^{\beta} w(x) f_m(x) f_n(x)\, dx = 0, \qquad m \ne n. \tag{8.3}$$

The set is **orthonormal** *if*

$$\int_{\alpha}^{\beta} w(x) f_m(x) f_n(x)\, dx = \begin{cases} 0, & \text{if } m \ne n; \\ 1, & \text{if } m = n. \end{cases} \tag{8.4}$$

Note that we can always obtain an orthonormal set from an orthogonal set by dividing each f_n by its **norm**

$$\|f_n\| = \left[\int_{\alpha}^{\beta} w(x) f_n^2(x)\, dx \right]^{1/2}.$$

Example 8.2. The set $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$ is orthogonal on $[-\pi, \pi]$ with respect to the weight function $w(x) \equiv 1$. Since $\|1\| = \sqrt{2\pi}$, $\|\sin nx\| = \|\cos nx\| = \sqrt{\pi}$, $n = 1, 2, \ldots$, then $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots \right\}$ is an orthonormal set.

Theorem 8.3 (Orthogonality). *Eigenfunctions which correspond to distinct eigenvalues of the regular S-L problem BVP (8.1), (8.2) are orthogonal wrt the weight function* $r(x)$ *on* $[\alpha, \beta]$.

Proof. Let λ, μ be distinct eigenvalues with corresponding eigenfunctions $\phi(x)$ and $\psi(x)$. Then

$$(p\phi')' + q\phi + \lambda r\phi = 0$$
$$(p\psi')' + q\psi + \mu r\psi = 0$$

Multiplying the first equation by ψ and the second by ϕ and subtracting gives

$$\psi(p\phi')' - \phi(p\psi')' + (\lambda - \mu) r\phi\psi = 0, \tag{8.5}$$

i.e.

$$\frac{d}{dx}\left[(p\phi')\psi - (p\psi')\phi\right] + (\lambda - \mu) r\phi\psi = 0, \tag{8.6}$$

i.e.

$$\frac{d}{dx}\left[p\left(\phi'\psi - \phi\psi'\right)\right] + (\lambda - \mu) r\phi\psi = 0. \tag{8.7}$$

Integrating from $x = \alpha$ to $x = \beta$ gives

$$\begin{aligned}(\lambda - \mu) \int_\alpha^\beta r(x)\phi(x)\psi(x)\,dx &= \left[p(x)\left(\phi'\psi - \phi\psi'\right)\right]_\alpha^\beta \\ &= \left[p(x)W(\phi,\psi)(x)\right]_\alpha^\beta \\ &= 0,\end{aligned}$$

for the BCs (8.2) give $W(\phi,\psi)(\alpha) = 0 = W(\phi,\psi)(\beta)$ ☐

Theorem 8.4 (Real eigenvalues)**.** *Then eigenvalues of the regular S-L BVP are real and have real-valued associated eigenfunctions.*

Proof. Suppose λ (possibly complex) is an eigenvalue with associated eigenfunction $\phi(x)$, so

$$(p\phi')' + q\phi + \lambda r\phi = 0.$$

Taking the complex conjugate gives

$$(p\overline{\phi}')' + q\overline{\phi} + \overline{\lambda} r\overline{\phi} = 0.$$

So $\overline{\lambda}$ is also an eigenvalue with associated eigenfunction $\overline{\phi}(x)$.

If $\lambda \neq \overline{\lambda}$, then, by Theorem 8.3

$$\int_\alpha^\beta r(x)\phi(x)\overline{\phi}(x)\,dx = \int_\alpha^\beta r(x)\left|\phi(x)\right|^2\,dx = 0.$$

Since $r(x) > 0$, the integrand is positive and the integral cannot be zero. Hence $\lambda = \overline{\lambda}$ so that λ is real.

Now if $\phi(x)$ is itself not real-valued then its real or imaginary part is itself an eigenfunction which is real-valued and corresponds to λ. ☐

We also state three further properties without proof.

Theorem 8.5. *Then eigenvalues of the regular S-L BVP (8.1), (8.2) form a countable, increasing sequence*

$$\lambda_1 < \lambda_2 < \dots, \qquad \textit{with} \lim_{n\to\infty} \lambda_n = \infty.$$

Theorem 8.6 (Oscillation)**.** *Then nth eigenfunction has $n-1$ zeros on the interval $\alpha < x < \beta$.*

Theorem 8.7 (Monotonicity)**.** *Reducing the interval (α,β), increasing $p(x)$, decreasing $q(x)$ or decreasing $r(x)$, increases all the eigenvalues.*

This theorem can sometimes be used to obtain information on the eigenvalues of a S-L BVP that cannot be solved exactly, as the following example shows.

Example 8.8. $[(1+x^2)\,y']' + \lambda y = 0, \quad 0 < x < 1, \qquad y(0) = y(1) = 0.$

8.2. **Eigenfunction expansions.** Fourier series is an example of an orthogonal set of functions which are used to expand functions.

Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be an orthogonal set of eigenfunctions with weight $r(x)$ on $\alpha \le x \le \beta$. Let $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$. Multiplying each term by $\phi_m(x)r(x)$ with m fixed and integrating over $[\alpha, \beta]$ we obtain

$$\begin{aligned}\int_{\alpha}^{\beta} r(x)f(x)\phi_m(x)\,dx &= \int_{\alpha}^{\beta} r(x)\sum_{n=1}^{\infty} c_n\phi_n(x)\phi_m(x)\,dx\\ &= \sum_{n=1}^{\infty} c_n \int_{\alpha}^{\beta} r(x)\phi_n(x)\phi_m(x)\,dx\\ &= c_m \int_{\alpha}^{\beta} r(x)\phi_m^2(x)\,dx\end{aligned}$$

giving

$$c_m = \frac{\int_{\alpha}^{\beta} r(x)f(x)\phi_m(x)\,dx}{\int_{\alpha}^{\beta} r(x)\phi_m^2(x)\,dx}, \quad m = 1, 2, \dots .$$

The above is a formal derivation of the coefficients and requires conditions for the term-by-term integration to be valid.

Theorem 8.9 (Eigenfunction expansion)**.** *Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be an orthonormal set of eigenfuntions for the S-L BVP (8.1) and (8.2) and let $f(x)$ be continuous on $[\alpha, \beta]$ with f' piecewise continuous. If $f(x)$ satisfies the BCs (8.2) then $f(x)$ is given by the expansion*

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad \alpha \le x \le \beta, \quad \textit{where} \quad c_n = \int_{\alpha}^{\beta} r(x)f(x)\phi_n(x)\,dx.$$

Moreover the expansion is uniformly convergent on $[\alpha, \beta]$.

Example 8.10. $y'' + \lambda y = 0, \quad 0 < x < \pi, \qquad y(0) = y(\pi) = 0.$ (See Example 6.2.)

8.3. **Nonhomogeneous BVPs: eigenfunction expansions.** Consider the BVP

$$(py')' + qy + \mu ry = -h(x), \quad \alpha \le x \le \beta \tag{8.8}$$

$$\begin{cases} a_1y(\alpha) + a_2y'(\alpha) = 0\\ b_1y(\beta) + b_2y'(\beta) = 0\end{cases} \tag{8.9}$$

where μ is a fixed real number. Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be the eigenfunctions of the associated (homogeneous) S-L BVP (8.1) and (8.2). We can write

$$y(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$$

for some constants c_n which must be determined. Thus, if

$$L[y] = (py')' + qy$$

then

$$\begin{aligned}
L[y] + \mu r y &= L\left[\sum_{n=1}^{\infty} c_n \phi_n(x)\right] + \mu r \sum_{n=1}^{\infty} c_n \phi_n(x) \\
&= \sum_{n=1}^{\infty} c_n L[\phi_n(x)] + \sum_{n=1}^{\infty} \mu r c_n \phi_n(x) \\
&= \sum_{n=1}^{\infty} (-c_n \lambda_n r \phi_n(x)) + \sum_{n=1}^{\infty} \mu r c_n \phi_n(x) \\
&= r \sum_{n=1}^{\infty} (\mu - \lambda_n) c_n \phi_n(x).
\end{aligned}$$

We can also expand the function h/r as

$$\frac{h(x)}{r(x)} = \sum_{n=1}^{\infty} \gamma_n \phi_n(x)$$

where

$$\gamma_n = \frac{\int_\alpha^\beta r \frac{h}{r} \phi_n \, dx}{\int_\alpha^\beta r \phi_n^2 \, dx} = \frac{\int_\alpha^\beta h(x) \phi_n(x) \, dx}{\int_\alpha^\beta r \phi_n^2(x) \, dx}.$$

Thus

$$h = r \sum_{n=1}^{\infty} \gamma_n \phi_n$$

and the differential equation becomes

$$r \sum_{n=1}^{\infty} (\mu - \lambda_n) c_n \phi_n = -r \sum_{n=1}^{\infty} \gamma_n \phi_n.$$

Since $r(x) > 0$ we obtain

$$\sum_{n=1}^{\infty} \left[c_n (\lambda_n - \mu) - \gamma_n\right] \phi_n(x) = 0.$$

This expansion sums to zero if and only if

$$c_n (\lambda_n - \mu) - \gamma_n = 0, \quad n = 1, 2, \ldots. \tag{8.10}$$

There are two cases to consider:

(1) $\mu \neq \lambda_n$ for any n: Then $c_n = \frac{\gamma_n}{\lambda_n - \mu}$ and

$$y(x) = \sum_{n=1}^{\infty} \frac{\gamma_n}{\lambda_n - \mu} \phi_n$$

is the unique formal (since we do not know if it converges) solution. (Unique because $L[y] + \mu r y = 0$ has no solutions.)

(2) $\mu = \lambda_N$ for some N. Then for (8.10) to be satisfied we must have $\gamma_N = 0$. If $\gamma_N \neq 0$, there is a contradiction and the BVP has no solution. If $\gamma_N = 0$, there is no restriction on c_N which means we have a one-parameter family of solutions. Note that when $\gamma_N = 0$ then $\int_\alpha^\beta h(x)\phi_N(x)\,dx = 0$, i.e. h/r and ϕ_N are orthogonal.

Theorem 8.11. *The formal solution of the nonhomogeneous BVP (8.8), (8.9) is given as an eigenfunction expansion of the eigenfunctions of the associated homogeneous S-L BVP by*

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where

(1) if μ is not an eigenvalue then $c_n = \gamma_n/(\lambda_n - \mu)$
(2) if $\mu = \lambda_N, \gamma_N = 0$, then $c_n = \gamma_n/(\lambda_n - \mu), n \neq N$ and c_N arbitrary

Example 8.12. $y'' = -f(x), \quad 0 < x < 1, \qquad y(0) = y(1) = 0$

8.4. **Green's function and eigenfunction expansions.**

Corollary 8.13 (of Theorem 8.11)**.** *The Green's function of the BVP (8.8), (8.9) is given in terms of the eigenfunctions of the associated S-L BVP (8.1), (8.2) by*

$$G(x,t) = \sum_{n=1}^{\infty} \frac{\phi_n(t)\phi_n(x)}{(\lambda_n - \mu)\nu_n}$$

where $\nu_n = \int_\alpha^\beta r\phi_n^2\,dx$ is the normalisation constant.

Proof. This is just a rewrite of the solution obtained in the proof of Theorem 8.11. We have the solution given by, on substituting for the coefficients c_n,

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\ &= \sum_{n=1}^{\infty} \left\{ \frac{\int h(t)\phi_n(t)\,dt}{(\lambda_n - \mu)\nu_n} \right\} \phi_n(x) \\ &= \int_\alpha^\beta \left\{ \sum_{n=1}^{\infty} \frac{\phi_n(t)\phi_n(x)}{(\lambda_n - \mu)\nu_n} \right\} h(t)\,dt \\ &= \int_\alpha^\beta G(x,t)h(t)\,dt. \end{aligned}$$

□

The following example links this section with the previous one.

Example 8.14.

$$y'' = -f(x), \quad 0 < x < 1, \qquad y(0) = y(1) = 0.$$

(See Example 7.4.)

8.5. **Self-adjoint operators.**

We use the notation

$$L[y] = (py')' + qy.$$

Then

$$\begin{aligned} uL[v] - vL[u] &= u(pv')' + uqv - v(pu')' - vqu \\ &= u(pv')' - v(pu')' \\ &= \frac{d}{dx}\left[p(uv' - vu')\right] \qquad \textbf{(Lagrange's identity)} \end{aligned}$$

So

$$\int_\alpha^\beta (uL[v] - vL[u])\ dx = [p(uv' - vu')]_\alpha^\beta \qquad \textbf{(Green's formula)}$$

If u and v satisfy the BCs (8.2) then Green's formula simplifies to

$$\int_\alpha^\beta (uL[v] - vL[u])\ dx = 0,$$

i.e.

$$\int_\alpha^\beta uL[v]\,dx = \int_\alpha^\beta vL[u]\,dx. \tag{8.11}$$

In terms of the inner product on the vector space $C[\alpha, \beta]$ defined by

$$(f, g) = \int_\alpha^\beta f(x)g(x)\,dx$$

the formula (8.11) reduces to

$$(L[u], v) = (u, L[v]). \tag{8.12}$$

A linear differential operator which satisfies (8.12) for all u, v in its domain is called a **self-adjoint operator**.

The S-L operator is therefore self-adjoint. This has important consequences: self-adjoint operators are like symmetric matrices in that their eigenvalues are real and their eigenfunctions orthogonal.

The eigenvalue problem becomes

$$L[y] = -\lambda ry.$$

9. LINEAR SYSTEMS

In this section we shall consider systems of equations of the form

$$\begin{cases} y_1'(x) = a_{11}(x)y_1(x) + a_{12}(x)y_2(x) + h_1(x) \\ y_2'(x) = a_{21}(x)y_1(x) + a_{22}(x)y_2(x) + h_2(x) \end{cases} \tag{9.1}$$

Many of the definitions and methods extends to n equations in n unknowns.

9.1. **Matrix formulation.** Let

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad \mathbf{h}(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix}.$$

Then the system (9.1) may be written in matrix form as

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x) + \mathbf{h}(x). \tag{9.2}$$

Example 9.1.

$$\begin{aligned} y_1' &= xy_1 + 2y_2 - e^x \\ y_2' &= x^2 y_1 - xy_2 + \sin x \end{aligned}$$

The second order DE

$$y''(x) + a(x)y'(x) + b(x)y(x) = g(x)$$

may be rewritten as a second order system. One way is as follows: Let $y_1 = y$, $y_2 = y'$. Then

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -b(x)y_1 - a(x)y_2 + g(x). \end{aligned}$$

Example 9.2.

$$y'' + 3y' + 2y = 4x.$$

Theorem 9.3 (Existence and uniqueness)**.** *If the functions $a_{ij}(x)$ and $h_i(x)$ are continuous on an interval I containing x_0 then the IVP*

$$\mathbf{y}'(x) = A\mathbf{y}(x) + \mathbf{h}(x), \quad \mathbf{y}(x_0) = \boldsymbol{\eta}_0$$

has a unique solution.

9.2. **Constant coefficient systems.**

9.2.1. *Homogeneous systems.*

$$\mathbf{y}'(x) = A\mathbf{y}(x), \qquad \text{where } A \text{ is a constant matrix.}$$

General solution.

Take $\mathbf{y}(x) = e^{\lambda x}\mathbf{u}$ as a trial solution, where $\mathbf{u}$ is a constant vector. Substituting into the DE gives $\lambda e^{\lambda x}\mathbf{u} = Ae^{\lambda x}\mathbf{u}$, so $\lambda\mathbf{u} = A\mathbf{u}$. Thus $\mathbf{y}(x)$ is a solution if λ is an eigenvalue of A and $\mathbf{u}$ the eigenvector.

Theorem 9.4 (General solution)**.** *If A has two linearly independent eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ corresponding to eigenvalues λ_1, λ_2 (not necessarily distinct) then the general solution of (9.2) is*

$$\mathbf{y}(x) = c_1 e^{\lambda_1 x}\mathbf{u}_1 + c_2 e^{\lambda_2 x}\mathbf{u}_2,$$

where c_1 and c_2 are arbitrary constants.

Corollary 9.5. *If A is real and the eigenvectors are complex $\alpha \pm i\beta$ then the general solution may be written*

$$\mathbf{y}(x) = c_1 \operatorname{Re}\left(e^{\lambda x}\mathbf{u}\right) + c_2 \operatorname{Im}\left(e^{\lambda x}\mathbf{u}\right).$$

Example 9.6.

$$\begin{aligned} y_1' &= 3y_1 + y_2 \\ y_2' &= 2y_1 + 4y_2 \end{aligned}$$

Note: the theorem may be extended to the case when A has only one eigenvector.

Canonical forms.

Consider the effect of a coordinate transformation on the system. Let $\mathbf{y} = P\mathbf{z}$, where P is a constant invertible matrix. Then $\mathbf{y}' = A\mathbf{y}$ may be rewritten as $P\mathbf{z}' = AP\mathbf{z}$ or as $\mathbf{z}' = J\mathbf{z}$, where $J = P^{-1}AP$.

Now P can be chosen so that J is one of only three types. Hence there are only three types of second order system to solve.

These **canonical forms**, J, are classified by the eigenvalues λ_1, λ_2 of A as follows, where $\mathbf{u}_1, \mathbf{u}_2$ are the corresponding eigenvectors:

(1) **real, distinct.**

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad P = (\mathbf{u}_1\mathbf{u}_2)$$

(2) **real, equal.**

(a) A diagonal.

$$J = A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$$

(b) A not diagonal.

$$J = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}, \qquad P = (\mathbf{u}\mathbf{v}), \quad \text{where} \quad (A - \lambda I)\mathbf{v} = \mathbf{u}$$

(3) **complex $\alpha \pm \beta i$.**

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad P = (\mathbf{u}_R\mathbf{u}_I), \quad \text{where} \quad \mathbf{u}_R = \mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{u}_I = (\mathbf{u}_1 - \mathbf{u}_2)/i.$$

Theorem 9.7 (Canonical forms)**.** *The system $\mathbf{y}' = A\mathbf{y}$ can be transformed into one of the three canonical forms $\mathbf{z}' = J\mathbf{z}$, where J is the Jordan normal form of A.*

Reminders:

- If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $P^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- Also $\sum \lambda = \lambda_1 + \lambda_2 = \operatorname{Tr} A, \quad \Pi\lambda = \lambda_1\lambda_2 = \det A$.

Initial value problems.

Example 9.8.

$$\mathbf{y}' = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

9.2.2. *Nonhomogeneous systems.*

$$\mathbf{y}'(x) = A\mathbf{y}(x) + \mathbf{h}(x). \tag{9.3}$$

General solution.

Theorem 9.9. *The general solution of (9.3) is given by*

$$\mathbf{y}(x) = \mathbf{y}_h(x) + \mathbf{y}_p(x),$$

where $\mathbf{y}_h(x)$ *is a (general) solution of the homogeneous equation* $\mathbf{y}' = A\mathbf{y}$ *and* $\mathbf{y}_p(x)$ *is a (particular) solution of the nonhomogeneous equation.*

Example 9.10.

$$\begin{aligned} y_1' &= 3y_1 - y_2 + e^{3x} \\ y_2' &= 2y_1 - 3e^{3x} \end{aligned}$$

Solution: The associated homogeneous problem was considered in Example 9.8 where the general solution was found to be

$$\mathbf{y}_h = c_1 e^{2x}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{x}\begin{pmatrix}1\\1\end{pmatrix}.$$

For the nonhomogeneous problem under consideration we take the trial solution $\mathbf{y}(x) = e^{3x}\mathbf{v}$, where $\mathbf{v}$ is a constant vector. A substitution into the DE yields

$$3e^{3x}\mathbf{v} = Ae^{3x}\mathbf{v} + e^{3x}\begin{pmatrix}1\\-3\end{pmatrix},$$

or equivalently

$$(3I - A)\mathbf{v} = \begin{pmatrix}1\\-3\end{pmatrix}.$$

Since $3I - A = \begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}$, we have

$$\mathbf{v} = (3I - A)^{-1}\begin{pmatrix}1\\-3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3 & -1\\2 & 0\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix},$$

and hence

$$\mathbf{y}_p = e^{3x}\begin{pmatrix}3\\1\end{pmatrix}$$

. So the general solution is

$$\mathbf{y}(x) = \mathbf{y}_h(x) + \mathbf{y}_p(x) = c_1 e^{2x}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{x}\begin{pmatrix}1\\1\end{pmatrix} + e^{3x}\begin{pmatrix}3\\1\end{pmatrix}.$$

Canonical form.

Let $\mathbf{y} = P\mathbf{z}$. Then (9.3) becomes $P\mathbf{z}' = AP\mathbf{z} + \mathbf{h}$, or, equivalently, $\mathbf{z}' = J\mathbf{z} + P^{-1}\mathbf{h}$, where $J = P^{-1}AP$.

Example 9.11.

$$\begin{aligned} y_1' &= 2y_1 + 3y_2 + e^{2x} \\ y_2' &= 2y_1 + y_2 + 4e^{2x} \end{aligned}$$

9.3. **General linear systems.**

9.3.1. *Homogeneous systems.*

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x). \tag{9.4}$$

Two vectors $\mathbf{y}_1 = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}$ are **linearly independent** if

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = 0 \quad \Longrightarrow \quad c_1 = c_2 = 0.$$

Let

$$M(x) = \begin{pmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{pmatrix} = (\mathbf{y}_1\mathbf{y}_2).$$

Then $\det M(x)$ is an analogue of the Wronskian, as the following theorem shows:

Theorem 9.12. *Let $\mathbf{y}_1, \mathbf{y}_2$ be solutions of (9.4). Then the following statements are equivalent.*

- $c_1\mathbf{y}_1 + c_2\mathbf{y}_2$ *is a general solution of (9.4)*
- $\det M(x_0) \neq 0$ *for some* $x_0 \in I$
- $\det M(x) \neq 0$ *for all* $x \in I$
- $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *are linearly independent*

The proof is as in §2.1.3.

Definition 9.13. *When $\mathbf{y}_1(x), \mathbf{y}_2(x)$ are linearly independent solutions of (9.4), then the matrix*

$$M(x) = (\mathbf{y}_1(x), \mathbf{y}_2(x))$$

is called a **fundamental matrix**. *If in addition $M(x_0) = I$, then $M(x)$ is called a* **state transition matrix** *and denoted by $M(x, x_0)$.*

Example 9.14.

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Theorem 9.15. *The general solution of $\mathbf{y}' = A\mathbf{y}$ is*

$$\mathbf{y}(x) = M(x)\mathbf{v},$$

where $\mathbf{v}$ is constant. The particular solution such that $\mathbf{y}(x_0) = \boldsymbol{\eta}_0$ is

$$\mathbf{y}(x) = M(x, x_0)\boldsymbol{\eta}_0.$$

Here are some further properties of the fundamental matrix $M(x)$.

1. (a) The fundamental matrix itself is a solution, i.e. $M'(x) = AM(x)$.
   (b) If $\det N(x) \neq 0$ and $N'(x) = AN(x)$, then $N(x)$ is a fundamental matrix.
2. (a) The matrix $N(x) = M(x)C$, where C is a non-singular constant matrix, is also a fundamental matrix. In particular $M(x, x_0) = M(x)\left[M(x_0)\right]^{-1}$.
   (b) If $M(x)$ and $N(x)$ are fundamental matrices then there exists a nonsingular constant matrix C such that $N(x) = M(x)C$.
3. **Abel's formula:**

$$\det M(x) = \det M(x_0) \exp\left\{\int_{x_0}^{x} \operatorname{Tr} A(s)\, ds\right\}.$$

(4) **Transition property** of transition matrices:

$$M(x_2, x_0) = M(x_2, x_1)M(x_1, x_0).$$

9.3.2. *Nonhomogeneous systems.*

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x) + \mathbf{h}(x). \tag{9.5}$$

Theorem 9.16. *The general solution of (9.5) is given by*

$$\mathbf{y}(x) = \mathbf{y}_h(x) + \mathbf{y}_p(x),$$

where $\mathbf{y}_h(x)$ is a (general) solution of the homogeneous equation $\mathbf{y}' = A(x)\mathbf{y}$ and $\mathbf{y}_p(x)$ is a (particular) solution of the nonhomogeneous equation.

In terms of a fundamental matrix, the solution is given as follows:

Theorem 9.17. *The general solution of (9.5) is*

$$\mathbf{y}(x) = M(x)\mathbf{v} + \int_{x_0}^{x} M(x, u)h(u)\,du,$$

where $M(x, u) = M(x)\,[M(u)]^{-1}$.

Proof. From Theorem 9.15, $\mathbf{y}_h(x) = M(x)\mathbf{v}$. Consider a particular integral $\mathbf{y}_p(x) = M(x)\mathbf{w}(x)$. Substituting into the DE gives $M'\mathbf{w} + M\mathbf{w}' = AM\mathbf{w} + \mathbf{h}$. So $M\mathbf{w}' = \mathbf{h}$, and so

$$\mathbf{w}'(x) = M^{-1}(x)\mathbf{h}(x).$$

Taking $\mathbf{w}(x_0) = 0$ then

$$\mathbf{w}(x) = \int_{x_0}^{x} M^{-1}(u)\mathbf{h}(u)\,du$$

so that

$$\begin{aligned}\mathbf{y}_p(x) &= M(x)\int_{x_0}^{x} M^{-1}(u)\mathbf{h}(u)\,du \\ &= \int_{x_0}^{x} M(x, u)\mathbf{h}(u)\,du\end{aligned}$$

□

Corollary 9.18. *If $\mathbf{y}_p(x_0) = \boldsymbol{\eta}_0$ then $\mathbf{v} = [M(x_0)]^{-1}\boldsymbol{\eta}_0$.*

Part II: PARTIAL DIFFERENTIAL EQUATIONS

We shall now consider equations of the form

$$L[u] = f$$

where L is a linear partial differential operator of first or second order in the variables.

The **order** of a PDE is the order of the highest derivative. A PDE is **linear** if it is algebraically linear in u, u_x, u_y, etc. For example

$$L[u] = u_x + yu_y = 0, \qquad \text{first order, linear}$$

$$L[u] = xu_{xx} + u_{yy} + yu_x = x^2y, \qquad \text{second order, linear}$$

The **principal of superposition** for a linear operator means that if $L[u] = f$ and $L[v] = g$, then $L[u + v] = f + g$.

A PDE has **arbitrary functions** in its solution.

10. PDEs: First-order linear equations

10.1. The constant coefficient equation.

First consider the PDE

$$au_x + bu_y = 0. \tag{10.1}$$

where $u = u(x, y)$, and a, b are constants which are not both zero.

Recall that the directional derivative of $f(x, y)$ in the direction of $\mathbf{v}$ is

$$\mathbf{v} \cdot \nabla f, \qquad \text{where} \quad \nabla f = f_x\mathbf{i} + f_y\mathbf{j}.$$

The PDE (10.1) asserts that

$$(a\mathbf{i} + b\mathbf{j}) \cdot \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right) = 0$$

i.e. that the rate of change of $u(x, y)$ in the direction of $a\mathbf{i} + b\mathbf{j}$ is zero.

Hence $u(x, y)$ is constant in this direction, i.e. on the straight lines

$$bx - ay = C \qquad (\textbf{characteristic curves})$$

So, on each characteristic, the value of u is a constant, and the constant only depends upon which characteristic we are on. Hence, u is a function of c, because as we change C we change characteristic. So

$$u(x, y) = f(C) = f(bx - ay). \tag{10.2}$$

Example 10.1. (**Transport equation**)

$$u_x + \gamma u_y = 0.$$

Next we consider the more general PDE

$$au_x + bu_y + cu = 0, \tag{10.3}$$

where a, b, c are constants.

Multiplying by an 'integrating factor' $e^{cx/a}$ gives

$$a\left[ue^{cx/a}\right]_x + b\left[ue^{cx/a}\right]_y = 0$$

which is just the previous equation (10.1) for the function $v(x, y) = e^{cx/a}u(x, y)$.

10.2. The variable coefficient equation.

$$a(x,y)u_x + b(x,y)u_y = 0. \tag{10.4}$$

As in the constant coefficient equation, the PDE asserts that

$$(a(x,y)\mathbf{i} + b(x,y)\mathbf{j}) \cdot \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right) = 0$$

so that $u(x,y)$ is constant along the **characteristic curves**

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \quad \Longrightarrow \quad \phi(x,y) = C \tag{10.5}$$

and the general solution is

$$u(x,y) = f(C) = f(\phi(x,y)). \tag{10.6}$$

Example 10.2. $u_x + yu_y = 0$.

Example 10.3. $yu_x + yu_y = 0, \qquad u(0,y) = e^{-y^2}$.

Example 10.4. $x^2u_x + (x^2+1)y^2u_y = 0, \qquad u(y,y) = 1/y$.

11. Second order linear PDES: classification

11.1. The constant coefficient equation.

Consider the general second order linear PDE

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y) \tag{11.1}$$

where a, b, c, d, e, f are constants.

Theorem 11.1. *By a linear transformation of the independent variables, equation (11.1) can be reduced to one of three* **canoncial forms***:*

1. **hyperbolic***: if* $b^2 - 4ac > 0$ *then (11.1) is reducible to*
$$u_{xy} + h(u_x, u_y, u) = G(x,y).$$
2. **parabolic***: if* $b^2 - 4ac = 0$ *then (11.1) is reducible to*
$$u_{xx} + h(u_x, u_y, u) = G(x,y).$$
3. **elliptic***: if* $b^2 - 4ac < 0$ *then (11.1) is reducible to*
$$u_{xx} + u_{yy} + h(u_x, u_y, u) = G(x,y).$$

Proof. Let $(x,y) \to (\xi,\eta)$ with $\xi = \alpha x + \beta y$ and $\eta = \gamma x + \delta y$, and $u(x,y) = \overline{u}(\xi,\eta)$. Then

$$\begin{aligned}
u_x &= \overline{u}_\xi \xi_x + \overline{u}_\eta \eta_x = \alpha\overline{u}_\xi + \gamma\overline{u}_\eta \\
u_y &= \overline{u}_\xi \xi_y + \overline{u}_\eta \eta_y = \beta\overline{u}_\xi + \delta\overline{u}_\eta \\
u_{xx} &= \alpha\left(\alpha\overline{u}_\xi + \gamma\overline{u}_\eta\right)_\xi + \gamma\left(\alpha\overline{u}_\xi + \gamma\overline{u}_\eta\right)_\eta = \alpha^2\overline{u}_{\xi\xi} + 2\alpha\gamma\overline{u}_{\xi\eta} + \gamma^2\overline{u}_{\eta\eta} \\
u_{xy} &= \alpha\left(\beta\overline{u}_\xi + \delta\overline{u}_\eta\right)_\xi + \gamma\left(\beta\overline{u}_\xi + \delta\overline{u}_\eta\right)_\eta = \alpha\beta\overline{u}_{\xi\xi} + (\alpha\delta + \beta\gamma)\,\overline{u}_{\xi\eta} + \gamma\delta\overline{u}_{\eta\eta} \\
u_{yy} &= \beta^2\overline{u}_{\xi\xi} + 2\beta\delta\overline{u}_{\xi\eta} + \delta^2\overline{u}_{\eta\eta}
\end{aligned}$$

So

$$\overline{L}[\overline{u}] = \left(a\alpha^2 + b\alpha\beta + c\beta^2\right)\overline{u}_{\xi\xi} + (2a\alpha\gamma + b(\alpha\delta + \beta\gamma) + 2c\beta\delta)\,\overline{u}_{\xi\eta} + \left(a\gamma^2 + b\gamma\delta + c\delta^2\right)\overline{u}_{\eta\eta} + \ldots = 0.$$

(1) $b^2 - 4ac > 0$: Choose $\frac{\alpha}{\beta}, \frac{\gamma}{\delta}$ as the distinct roots of the quadratic equation $ap^2 + bp + c$. Then the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ are zero, so that the PDE reduces to

$$L[\overline{u}] = (\text{const})\overline{u}_{\xi\eta} + h(\overline{u}_\xi, \overline{u}_\eta, \overline{u}) = \overline{G}(\xi, \eta)$$

(2) $b^2 - 4ac = 0$: Choose $\frac{\gamma}{\delta}$ as the root (repeated) of $ap^2 + bp + c$. Then the coefficient of $\overline{u}_{\eta\eta}$ is zero. Also, $\frac{\gamma}{\delta} = -\frac{b}{2a}$ implies that the coefficient of $\overline{u}_{\xi\eta}$ is zero. Then

$$L[\overline{u}] = (\text{const})\overline{u}_{\xi\xi} + h(\overline{u}_\xi, \overline{u}_\eta, \overline{u}) = \overline{G}(\xi, \eta).$$

(3) $b^2 - 4ac < 0$: The real transformation

$$\xi = \frac{2ay - bx}{\sqrt{4ac - b^2}}, \quad \eta = x$$

reduces the PDE to

$$L[\overline{u}] = (\text{const})\,(\overline{u}_{\xi\xi} + \overline{u}_{\eta\eta}) + h(\overline{u}_\xi, \overline{u}_\eta, \overline{u}) = \overline{G}(\xi, \eta).$$

☐

Classification: PDES are classified according to the sign of $b^2 - 4ac$ as hyperbolic, parabolic or elliptic as above.

Characteristics: The curves

$$\xi = \alpha x + \beta y = \text{constant}$$
$$\eta = \gamma x + \delta y = \text{constant}$$

are called the characteristics of the PDE. Their gradients $-\alpha/\beta$, $-\gamma/\delta$ are given by the roots of the quadratic equation

$$a\lambda^2 - b\lambda + c = 0.$$

Hence the characteristics are the solutions of the DE

$$a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0.$$

Canonical forms: Using the **characteristic coordinates** (ξ, η), the equation (11.1) reduces to one of the canonical forms listed in Theorem 11.1.

For the hyperbolic equation, a further change of coordinates transforms (11.1) into the form

$$u_{xx} - u_{xy} + h(u_x, u_y, u) = G(x, y).$$

This is sometimes called the second canonical form, and is usually the form used for the wave equation which we study later.

Example 11.2.

$$5u_{xx} - 4u_{xy} - u_{yy} = 0.$$

Example 11.3.

$$u_{xx} - 4u_{xy} + 4u_{yy} = e^y.$$

11.2. The variable coefficient equation.

Consider the general second order equation

$$a(x,y)u_{xx}+b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y). \tag{11.2}$$

The previous analysis for the constant coefficient equation (11.1) extends to the variable coefficient case as follows:

Definition 11.4. *The PDE (11.2) is classified as follows:*

1. *if* $b^2-4ac>0$ *then (11.2) is* **hyperbolic**
2. *if* $b^2-4ac=0$ *then (11.2) is* **parabolic**
3. *if* $b^2-4ac<0$ *then (11.2) is* **elliptic**

Since b^2-4ac is a function of x and y, the PDE may be hyperbolic in one region of the (x,y)-plane and elliptic in another.

Definition 11.5. *The* **characteristics** *of the PDE (11.2) are the solution curves of the ODE*

$$a(x,y)\left(\frac{dy}{dx}\right)^2-b(x,y)\frac{dy}{dx}+c(x,y)=0. \tag{11.3}$$

Theorem 11.6. *By transforming to* **characteristic coordinates***, equation (11.2) may be transformed to canonical form as in Theorem 11.1.*

For hyperbolic and parabolic equations, the **characteristic coordinates** are defined as follows:

1. **hyperbolic**: $b^2-4ac>0 \implies$ two families of characteristics (from (11.3)):
$$\phi(x,y)=C_1, \quad \psi(x,y)=C_2$$
and so characteristic coordinates
$$\xi=\phi(x,y), \quad \eta=\psi(x,y)$$
2. **parabolic**: $b^2-4ac=0 \implies$ one family of characteristics (from (11.3)):
$$\phi(x,y)=C$$
and so characteristic coordinates
$$\xi=\phi(x,y), \quad \eta=x \qquad (\eta \text{ arbitrary})$$

Example 11.7.

$$yu_{xx}-u_{yy}=0.$$

Example 11.8.

$$e^{2x}u_{xx}+2e^{x+y}u_{xy}+e^{2y}u_{yy}=0.$$

11.3. Standard equations.

The fact that any second order PDE may be put into canonical form is useful in that it enables us to put our efforts into studying the archetypal hyperbolic, parabolic and elliptic equations given by:

- **wave equation**:
$$u_{tt}-c^2u_{xx}=0$$
- **diffusion equation**:
$$u_t=ku_{xx}$$
- **Laplace's equation**:
$$u_{xx}+u_{yy}=0$$

12. THE ONE-DIMENSIONAL WAVE EQUATION

12.1. **The Cauchy problem.** Consider the wave equation on an infinite domain.

Theorem 12.1. *The solution of the IVP*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x), \qquad -\infty < x < \infty \\ u_t(x,0) &= g(x), \qquad -\infty < x < \infty \end{aligned}$$

where $f \in C^2(\mathbf{R})$, $g \in C^1(\mathbf{R})$, is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \qquad \text{(D'Alembert solution)} \tag{12.1}$$

Proof. **Characteristic coordinates**:

$$\text{characteristics} \quad \left(\frac{dx}{dt}\right)^2 - c^2 = 0 \quad \Longrightarrow \quad \frac{dx}{dt} = \pm c \quad \Longrightarrow \quad x = \pm ct + k$$

So

$$\xi = x + ct, \quad \eta = x - ct.$$

Transform PDE:

$$\begin{aligned} u_t &= \overline{u}_\xi \xi_t + \overline{u}_\eta \eta_t = c\overline{u}_\xi - c\overline{u}_\eta \\ u_{tt} &= c^2\overline{u}_{\xi\xi} - 2c^2\overline{u}_{\xi\eta} + c^2\overline{u}_{\eta\eta} \\ u_x &= \overline{u}_\xi \xi_x + \overline{u}_\eta \eta_x = \overline{u}_\xi + \overline{u}_\eta \\ u_{xx} &= \overline{u}_{\xi\xi} + 2\overline{u}_{\xi\eta} + \overline{u}_{\eta\eta} \end{aligned}$$

So the PDE transforms to $-4c^2\overline{u}_{\xi\eta} = 0$, i.e.

$$\overline{u}_{\xi\eta} = 0.$$

General solution:

$$\begin{aligned} \overline{u}(\xi,\eta) &= p(\xi) + q(\eta) \qquad p, q \text{ arbitrary} \\ u(x,t) &= p(x+ct) + q(x-ct) \end{aligned}$$

Initial conditions:

$$\begin{aligned} u(x,0) &= p(x) + q(x) = f(x) \\ u_t(x,0) &= cp'(x) - cq'(x) = g(x) \end{aligned}$$

Differentiating the first equation gives

$$2cp'(x) = cf'(x) + g(x).$$

Integrating then gives

$$p(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + k.$$

Then from the first equation

$$q(x) = f(x) - p(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - k.$$

Thus

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{1}{2}(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \end{aligned}$$

□

Example 12.2.

$$u_{tt} - c^2 u_{xx} = 0, \qquad f(x) = \sin x, \quad g(x) = 0.$$

Solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] \\ &= \sin x \cos ct. \end{aligned}$$

Example 12.3.

$$u_{tt} - c^2 u_{xx} = 0, \qquad f(x) = \sin x, \quad g(x) = e^x.$$

Solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} e^s\,ds \\ &= \sin x \cos ct + \frac{1}{2c}\left(e^{x+ct} - e^{x-ct}\right). \end{aligned}$$

General solution:

$$u(x,t) = p(x+ct) + q(x-ct)$$

is the sum of two waves

$p(x+ct)$	travelling to left with speed c
$q(x-ct)$	travelling to right with speed c

Domain of dependence:

The solution at $(\overline{x}, \overline{t})$ depends on the values of $g(x)$ on the interval $[A, B]$ and of $f(x)$ at endpoints A, B, where $A = \overline{x} - c\overline{t}$ and $B = \overline{x} + c\overline{t}$.

12.2. **Initial-boundary value problem.** Consider the wave equation on the bounded domain $0 < x < L$. The d'Alembert solution only determines $u(x,t)$ at point in the triangular region bounded by the x-axis and the characteristics $x + ct = 0$ and $x - ct = L$. [DIAGRAM?] Further information is needed in the form of boundary conditions at the ends $x = 0$ and $x = L$ of the interval.

Theorem 12.4. *The solution of the initial-boundary value problem*

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < L, \quad t > 0 \\ u(x,0) &= f(x), \qquad 0 \le x \le L \\ u_t(x,0) &= g(x), \qquad 0 \le x \le L \\ u(0,t) &= 0, \qquad t > 0 \\ u(L,t) &= 0, \qquad t > 0 \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds, \tag{12.2}$$

where F, G are period $2L$ odd extensions of f, g.

Proof. Clearly (12.2) satisfies the PDE and the initial conditions, leaving only the boundary conditions to be proved.

When $x = 0$:

$$u(0,t) = \frac{1}{2}\left[F(ct) + F(-ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} G(s)\,ds$$

is zero since F and G are odd functions.

When $x = L$:

$$u(L,t) = \frac{1}{2}\left[F(L+ct) + F(L-ct)\right] + \frac{1}{2c}\int_{L-ct}^{L+ct} G(s)\,ds.$$

Now since F is odd and of period $2L$,

$$F(L+\alpha) = -F(-L-\alpha) = -F(L-\alpha).$$

So F and likewise G are 'odd' functions about $x = L$. Hence $u(L,t) = 0$. ☐

Example 12.5. Let $f(x) = x(1-x)$, $g(x) = \sin^2 \pi x$, $c = L = 1$. Find $u\left(\frac{1}{2}, \frac{3}{4}\right)$.

Solution:

$$u\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{2}\left[F\left(\frac{5}{4}\right) + F\left(\frac{-1}{4}\right)\right] + \frac{1}{2}\int_{-1/4}^{5/4} G(s)\,ds.$$

$$F\left(\frac{5}{4}\right) = F\left(\frac{5}{4} - 2\right) = F\left(\frac{-3}{4}\right) = -F\left(\frac{3}{4}\right) = -f\left(\frac{3}{4}\right) = -\frac{3}{4}\cdot\frac{1}{4} = -\frac{3}{16}.$$

$$F\left(\frac{-1}{4}\right) = -F\left(\frac{1}{4}\right) = -f\left(\frac{1}{4}\right) = -\frac{1}{4}\cdot\frac{3}{4} = -\frac{3}{16}.$$

$$\begin{aligned}\int_{-1/4}^{5/4} G(s)\,ds &= \int_{-1/4}^{1/4} + \int_{1/4}^{3/4} + \int_{3/4}^{5/4} = 0 + \int_{1/4}^{3/4} g(s)\,ds + 0 = \int_{1/4}^{3/4} \sin^2 \pi s\,ds \\ &= \left[\frac{1}{2}\left(s - \frac{1}{2\pi}\sin 2\pi s\right)\right]_{1/4}^{3/4} = \frac{1}{4} + \frac{1}{2\pi}.\end{aligned}$$

$$u\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{2}\left[-\frac{3}{16} - \frac{3}{16}\right] + \frac{1}{2}\left[\frac{1}{4} + \frac{1}{2\pi}\right] = \frac{1}{4\pi} - \frac{1}{16}.$$

[CHECK alan's page 70 reverse] The d'Alembert extension is not a useful solution for evaluating the solution except at isolated points, but it does give insight into the effect of boundaries.

A more effective method for obtaining an exact solution is that of **separation of variables**.

13. The diffusion equation (heat equation)

13.1. **Unbounded domain.** Consider the IVP

$$\begin{aligned} u_t &= k u_{xx}, \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x), \qquad -\infty < x < \infty \end{aligned}$$

Using Fourier transforms, the solution is obtained as

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) \exp\left\{ -\frac{(x-y)^2}{4kt} \right\} dy. \tag{13.1}$$

[SEE ALAN's transparency]

Since for each $t > 0$, $\exp\left\{-\frac{(x-y)^2}{4kt}\right\} > 0$ for all x, y, it follows from (13.1) that for each $t > 0$ and all x, the value of $u(x,t)$ is influenced by every value of $f(y)$, $-\infty < y < \infty$.

So the basic property of diffusion is that the initial disturbance spreads out in a smooth function. The speed of propagation is infinite and the solution spreads out to infinity immediately.

If the diffusion equation is defined on a bounded domain ($0 < x < L, t > 0$) then boundary contitions are required at $x = 0$ and $x = L$. As with the wave equation, the method of separation of variables is often an effective method of obtaining an exact solution.

13.2. **Bounded domain: initial-boundary value problem.** Consider the problem

$$\begin{aligned} u_t &= k u_{xx}, \qquad 0 < x < L, \quad t > 0 \\ u(x,0) &= f(x), \qquad 0 \le x \le L \\ u(0,t) &= 0, \qquad t > 0 \\ u(L,t) &= 0, \qquad t > 0 \end{aligned}$$

Let

$$u(x,t) = X(x)T(t).$$

Then the PDE implies

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = \lambda$$

and the boundary conditions imply

$$X(0) = X(L) = 0.$$

For $X(x)$ we have the S-L BVP

$$X'' - \lambda X = 0, \qquad X(0) = X(L) = 0.$$

It has eigenvalues and eigenfunctions

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots.$$

For $T(t)$ we have

$$T' + \left(\frac{n\pi}{L}\right)^2 kT = 0 \quad \Longrightarrow \quad T_n(t) = \exp\left\{-\left(\frac{n\pi}{L}\right)^2 kt\right\}.$$

By superposition:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left\{-\left(\frac{n\pi}{L}\right)^2 kt\right\} \tag{13.2}$$

The b_n are obtained from the initial condition:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad \Longrightarrow \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

The rapidly decaying exponential factor ensures convergence of the formal solution.

14. Hyperbolic and parabolic equations

14.1. **Parabolic equations.** The behaviour of the solution of the heat equation is typical of parabolic equations generally. That is, a change in the initial data at one point influences the value of $u(x,t)$ at every point – infinite speed of propagation.

14.2. **Hyperbolic equations.** The behaviour of the solution of the wave equation is typical of hyperbolic equations generally. That is, the initial data is propagated along characteristics – finite speed of propagation.

Example 14.1.

$$yu_{xx} - u_{yy} = 0, \qquad y > 0.$$

From Example 11.7, the **characteristics** are

$$\frac{2}{3}y^{3/2} - x = C_1, \qquad \frac{2}{3}y^{3/2} + x = C_2.$$

Region of influence: (e.g. data values at $x = 4$.) The characteristics through $(4,0)$ are $\frac{2}{3}y^{3/2} - x = -4$ and $\frac{2}{3}y^{3/2} + x = 4$, so the region of influence is bounded by these curves.
Domain of dependence: (e.g. the value $u(4,9)$.) The characteristics through $(4,9)$ are $\frac{2}{3}y^{3/2} - x = 18 - 4 = 14$ and $\frac{2}{3}y^{3/2} + x = 18 + 4 = 22$. These characteristics intersect the data line $y = 0$ at $x = -14$ and $x = 22$. So the domain of dependence of $u(4,9)$ is the interval $-14 \le x \le 22$.

15. Nonhomogeneous problems

15.1. **Time independent nonhomogeneous problems.** One approach is to reduce the problem to a homogeneous one. We shall outline the method for the diffusion equation, but note that it applies to other PDEs.

The nonhomogeneous problem is

$$\begin{aligned} u_t &= ku_{xx} + h(x), \qquad 0 < x < L, \quad t > 0 \\ u(x,0) &= f(x), \qquad 0 \le x \le L \\ u(0,t) &= p, \qquad t > 0 \quad (p \text{ constant}) \\ u(L,t) &= q, \qquad t > 0 \quad (q \text{ constant}) \end{aligned}$$

The corresponding time independent problem (steady state $u_t \equiv 0$) is

$$\begin{aligned} kU_{xx} + h(x) &= 0, \qquad 0 < x < L \\ U(0) &= p \\ U(L) &= q \end{aligned}$$

Putting $w(x,t) = u(x,t) - U(x)$ we obtain the PDE

$$w_t + U_t = kw_{xx} + kU_{xx} + h(x)$$

with BCs

$$\begin{aligned} w(0,t) &= u(0,t) - U(0) = p - p = 0 \\ w(L,t) &= u(L,t) - U(L) = q - q = 0 \end{aligned}$$

and IC

$$w(x,0) = u(x,0) - U(x) = f(x) - U(x).$$

Hence the solution is given by

$$u(x,t) = w(x,t) + U(x)$$

where $U(x)$ is a solution of the time independent problem and $w(x,t)$ satisfies the **homogeneous problem**

$$\begin{aligned} w_t &= kw_{xx}, \qquad 0 < x < L, \quad t > 0 \\ w(x,0) &= f(x) - U(x), \qquad 0 \le x \le L \\ w(0,t) &= 0, \qquad t > 0 \\ w(L,t) &= 0, \qquad t > 0 \end{aligned}$$

This problem can be solved using the method of separation of variables. The method extends to Neumann and Robin boundary conditions.

Example 15.1.

$$\begin{aligned} &u_t = u_{xx} + x^2, \qquad 0 < x < \pi, \quad t > 0 \\ &u(x,0) = x, \qquad 0 \le x \le \pi \\ &u(0,t) = 0, \qquad t > 0 \\ &u(\pi,t) = \pi, \qquad t > 0 \end{aligned}$$

Time-independent problem:

$$U_{xx} = -x^2, \qquad U(0) = 0, \quad U(\pi) = \pi.$$

Integrating twice:

$$U = -\frac{1}{12}x^4 + Ax + B.$$

Initial conditions imply

$$U(x) = \frac{1}{12}x\left(\pi^3 - x^3\right) + x.$$

Homogeneous problem:

Let $u(x,t) = w(x,t) + U(x)$. Then

$$\begin{aligned} w_t &= w_{xx}, \qquad 0 < x < \pi, \quad t > 0 \\ w(x,0) &= x - U(x) = -\frac{1}{12}x\left(\pi^3 - x^3\right) \\ u(0,t) &= 0, \qquad t > 0 \\ u(\pi,t) &= \pi, \qquad t > 0 \end{aligned}$$

Method of separation of variables gives (as in §13.2)

$$W(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx \qquad (L = 1 = k)$$

with

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} W(x,0)\sin nx\, dx \\ &= \frac{2}{\pi}\int_0^{\pi}\left\{-\frac{1}{12}x\left(\pi^3 - x^3\right)\right\}\sin nx\, dx \\ &= \frac{2\pi(-1)^n}{\pi^2} + \frac{4\left(1-(-1)^n\right)}{\pi n^5}. \end{aligned}$$

Hence the **solution** is

$$\begin{aligned} u(x,t) &= w(x,t) + U(x) \\ &= \sum_{n=1}^{\infty}\left\{-\frac{1}{12}x\left(\pi^3 - x^3\right)\right\} e^{-n^2 t}\sin x + \frac{1}{12}x(\pi^3 - x^3) + x. \end{aligned}$$

15.2. **Time dependent problems.**

Example 15.2.

$$\begin{aligned} u_t - u_{xx} &= h(x,t), \qquad 0 < x < \pi, \quad t > 0 \\ u(x,0) &= f(x), \qquad 0 \le x \le \pi \\ u(0,t) &= 0, \qquad t > 0 \\ u(\pi,t) &= \pi, \qquad t > 0 \end{aligned}$$

Example 15.3.

$$\begin{aligned} u_t - c^2 u_{xx} &= xt, \qquad 0 < x < \pi, \quad t > 0 \\ u(x,0) &= \sin x, \qquad 0 \le x \le \pi \\ u_t(x,0) &= 0, \qquad 0 \le x \le \pi \\ u(0,t) &= 0, \qquad t > 0 \\ u(\pi,t) &= \pi, \qquad t > 0 \end{aligned}$$

Solution. First find the appropriate eigenfunction basis by solving the homogeneous problem

$$u_{tt} - c^2 u_{xx} = 0, \qquad u(0,t) = u(\pi,t) = 0.$$

Let $u(x,t) = X(x)T(t)$ be a solution. Then plugging into the DE, we get $XT'' - c^2X''T = 0$, so $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda$, where λ is a constant. The BCs then imply $U(0,t) = X(0)T(t) = 0 \implies$

$X(0) = 0$ and $U(\pi, t) = X(\pi)T(t) = 0 \implies X(\pi) = 0$. So we have a S-L eigenvalue problem for $X(x)$:

$$X'' + \lambda X = 0, \qquad X(0) = X(\pi) = 0.$$

Solving gives eigenvalues $\lambda_n = n^2$ and eigenfunctions $X_n(x) = \sin nx$. For this example let us work with the normalised eigenfunctions $\hat{X}_n(x) = \sqrt{2/\pi}\sin nx$.

Now write a solution of the original nonhomogeneous problem as an eigenfunction expansion:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{X}_n(x).$$

It remains to determine the coefficients $b_n(t)$.

Substitution into the PDE gives

$$\sum_{n=1}^{\infty} b_n{}''(t)\hat{X}_n(x) - c^2\sum_{n=1}^{\infty} b_n(t)(-n^2)\hat{X}_n(x) = \sum_{n=1}^{\infty}\gamma_n(t)\hat{X}_n(x),$$

where the right-hand side is the eigenfunction expansion of xt, so that

$$\gamma_n(t) = \int_0^{\pi} xt\hat{X}_n(x)\,dx = \frac{\sqrt{2\pi}(-1)^{n+1}}{n}t.$$

Then equating coefficients gives

$$b_n''(t) + n^2c^2b_n(t) = \gamma_n(t) = \frac{\sqrt{2\pi}(-1)^{n+1}}{n}t.$$

Then initial conditions imply:

$$u(x,0) = \sum_{n=1}^{\infty} b_n(0)\sqrt{2/\pi}\sin nx = \sin x \qquad \implies \qquad b_1(0) = \sqrt{\pi/2}, \quad b_n(0) = 0, \quad n \geq 2.$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n'(0)\sqrt{2/\pi}\sin nx = 0 \qquad \implies \qquad b_n'(0) = 0, \quad n \geq 1.$$

So we arrive at the second-order ODEs

$$b_n'' + n^2c^2b_n = \frac{\sqrt{2\pi}(-1)^{n+1}}{n}t.$$

The general solutions are

$$b_n(t) = A_n\cos nct + B_n\sin nct + \frac{\sqrt{2\pi}(-1)^{n+1}}{n^3c^2}t.$$

For $n = 1$ the initial conditions give $b_1(0) = A_1 = \sqrt{\pi/2}$ and $b_1'(0) = cB_1 + \frac{\sqrt{2\pi}}{c^2} = 0 \implies B_1 = -\frac{\sqrt{2\pi}}{c^3}$. Hence

$$b_1(t) = \sqrt{\pi/2}\cos ct - \frac{\sqrt{2\pi}}{c^3}\sin ct + \frac{\sqrt{2\pi}}{c^3}t.$$

For $n \geq 2$, initial conditions imply $b_n(0) = A_n = 0$ and $b'_n(0) = ncB_n + \frac{\sqrt{2\pi}(-1)^{n+1}}{n^3c^2} = 0 \implies B_n = \frac{\sqrt{2\pi}(-1)^n}{n^4c^2}$. So

$$b_n(t) = \frac{\sqrt{2\pi}(-1)^n}{n^4c^2}\sin nct + \frac{\sqrt{2\pi}(-1)^{n+1}}{n^3c^2}t.$$