Differential Equations (MA3603, 1st teaching term) -- Exercises 2

Second order equations: nonhomogeneous

1. Use the method of variation of parameters to obtain a particular integral of the equation

$$y'' + 4y = \sec 2x, \quad -\pi/4 < x < \pi/4.$$

Hence find the general solution of the equation.

2. Consider the initial value problem

$$4x^2y'' + 9xy' + y = \sqrt{x}, \quad x > 0,.$$
$$y(1) = 1, \quad y'(1) = -1.$$

(a) Find a fundamental set of solutions for the associated homogeneous Cauchy-Euler equation by using a trial solution of the form $y(x) = x^r$.
(b) Find a particular integral by using the method of variation of parameters.
(c) Write down the general solution and satisfy the initial conditions.

3. Using Theorem 2.8, solve the initial value problems:

(i) $y'' - 3y' + 2y = f(x), \quad y(0) = 0, \; y'(0) = 0,$

(ii) $y'' - 3y' + 2y = f(x), \quad y(0) = 1, \; y'(0) = -1.$

4. (*This question is bookwork which was not covered in the lectures.*)

Let p, q be continuous functions on $I \subseteq \mathbf{R}$. Show that the set, V, of twice differentiable functions which satisfy

$$L[\varphi] = \varphi'' + p\varphi' + q\varphi = 0$$

forms a vector space under the usual addition and scalar multiplication for functions. Explain why a fundamental set of solutions of $L[\varphi] = 0$ is a basis for this space.

Higher order equations

5. Verify by substitution that $\sin(x^2)$ is a particular integral of the equation

$$\frac{d^4y}{dx^4} - y = (16x^4 - 13)\sin(x^2) - 48x^2\cos(x^2).$$

Find the general solution.

6. Show that $\{1, e^{-x}, e^{3x}\}$ is a fundamental set of solutions of the equation

$$y''' - 2y'' - 3y' = 0 .$$

Hence obtain a fundamental set of solutions $\{y_0, y_1, y_2\}$ which satisfies the conditions

$$\left.\frac{d^j y_i}{dx^j}\right|_{x=0} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i,j = 0,1,2.$$

Express the solution which satisfies the initial conditions

$$y(0) = \alpha_0, \; y'(0) = \alpha_1, \; y''(0) = \alpha_2$$

as a linear combination of y_0, y_1 and y_2.