## MA3606 DIFFERENTIAL EQUATIONS - EXERCISES 4

**Nonhomogeneous boundary-value problems: Green's function**

1. Find the Green's function for the nonhomogeneous boundary-value problem

$$y'' = -f(x), \qquad 0 < x < 1,$$

$$y(0) = 0, \quad y(1) + y'(1) = 0.$$

Hence solve the problem when $f(x) = x$.

3. Consider the nonhomogeneous boundary-value problem

$$y'' + k^2 y = -f(x) \qquad 0 < x < 1, \quad k \in \mathbf{R},$$

$$y'(0) = y(1) = 0.$$

(i) Show that the problem has a unique solution if $k \neq (n + \frac{1}{2})\pi$, $n \in \mathbf{Z}$.
(ii) Find the Green's function for the problem.
(iii) Use the Green's function to find the solution when $f(x) = 1$.

3. Find the Green's function $G(x,t)$ for the problem

$$xy'' + y' = -f(x), \qquad 1 < x < 2,$$

$$y(1) - y'(1) = 0, \quad y'(2) = 0.$$

Verify that $y(x) = \int_1^2 G(x,t) f(t)\, dt$ is a solution and prove that it is the only solution.

**Nonhomogeneous boundary-value problems: eigenfunction expansions**

4. Consider the nonhomogeneous Dirichlet problem

$$y'' + 3y = -x, \qquad 0 < x < 1,$$

$$y(0) = y(1) = 0.$$

The eigenvalues and eigenfunctions of the associated homogeneous problem

$$y'' + \lambda y = 0, \qquad 0 < x < 1,$$

$$y(0) = y(1) = 0,$$

are given by $\lambda_n = n^2\pi^2$ and $\varphi_n(x) = \sin(n\pi x)$, $n = 1, 2, \ldots$ .

(i) Use the method of eigenfunction expansions to find the unique formal solution of the nonhomogeneous problem.
(ii) Find the Green's function of the problem as an eigenfunction expansion.