X3

DIFFERENTIAL EQUATIONS

Solutions to Exercise Sheet 1

1. (i) $\mu(x) = e^{\int 1\,dx} = e^x$ $\quad\therefore (e^x y)' = e^{2x}$ $\quad\therefore y = \frac{1}{2}e^x + \frac{1}{2}ce^{-x}$

$y(0) = \frac{1}{2} + \frac{1}{2}c = 0$ gives $c = -1$ $\quad\therefore y = \frac{1}{2}(e^x - e^{-x})$

(ii) $\mu(x) = e^{\int 2x\,dx} = e^{x^2}$ $\quad\therefore (e^{x^2}y)' = xe^{x^2}$ $\quad\therefore y = \frac{1}{2} + ce^{-x^2}$

$y(0) = \frac{1}{2} + c = 0$ gives $c = -\frac{1}{2}$ $\quad\therefore y = \frac{1}{2}(1 - e^{-x^2})$

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2. Separating the variables gives $\frac{y'}{\sqrt{1-y^2}} = x$ and integrating gives $\int \frac{dy}{\sqrt{1-y^2}} = \int x\,dx$.

So $\sin^{-1}y = \frac{1}{2}x^2 + C$ giving $y(x) = \sin(\frac{1}{2}x^2 + C)$, where C is a constant

$y(0) = 1 \Rightarrow C = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$. Choosing $n=0$, as the solution curve for each n is the same, we have that

$$y(x) = \sin\left(\tfrac{1}{2}(x^2 + \pi)\right)$$

By substitution, $y(x) = 1$ satisfies the d.e. and the i.c.

In this problem, $f(x,y) = x\sqrt{1-y^2}$ so $\frac{\partial f}{\partial y} = \frac{-xy}{\sqrt{1-y^2}}$.

Consequently, $\frac{\partial f}{\partial y}$ is not continuous at $y=1$, hence Picard's theorem cannot be applied.

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3. (a) Integrating the d.e. from x_0 to x and using the initial condition $y(0) = 0$ gives

$$y(x) = \int_0^x (1 + s\,y(s)^2)\,ds$$

(b) Let

$$y_{n+1}(x) = \int_0^x \left(1 + s\,(y_n(s))^2\right) ds$$

Taking $y_0 = 0$, then

$$y_1(x) = \int_0^x 1\,ds = x$$

$$y_2(x) = \int_0^x (1 + s.s^2)\,ds = \left[s + \tfrac{1}{4}s^4\right]_0^x = x + \tfrac{1}{4}x^4$$

$$y_3(x) = \int_0^x \left(1 + s\,(s + \tfrac{1}{4}s^4)^2\right) ds$$

$$= \int_0^x \left(1 + s^3 + \tfrac{1}{2}s^6 + \tfrac{1}{16}s^9\right) ds = x + \tfrac{1}{4}x^4 + \tfrac{1}{14}x^7 + \tfrac{1}{160}x^{10}$$

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4. (i) $p(x) = \frac{x}{(1+x^2)}$, $q(x) = -\frac{1}{(1+x^2)}$, $g(x) = \frac{\tan x}{(1+x^2)}$, $x_0 = 0$

$p(x)$, $q(x)$ continuous on $\mathbb{R}$, $g(x)$ discontinuous at $x = (n + \frac{1}{2})\pi$

$\therefore$ solution exists on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ containing $x_0 = 0$

(ii) unique solution exists on the interval $0 < x < \sqrt{3}$

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5. (i) Trial solution $y = e^{mx}$. Substitution in the o.d.e. gives $(m^2 + 5m + 4)e^{mx} = 0$.
Then $e^{mx} \neq 0$ gives $m^2 + 5m + 4 = 0$, so $m = -1, -4$. Hence $y_1 = e^{-x}$, $y_2 = e^{-4x}$ are solutions of the o.d.e.

Now $W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & e^{-4x} \\ -e^{-x} & -4e^{-4x} \end{vmatrix} = -3e^{-5x} \neq 0$ for any $x \in \mathbb{R}$

Hence $\{e^{-x}, e^{-4x}\}$ is a fundamental set of solutions.

(ii) $m = -2 \pm i$; $y_1 = e^{-2x}\cos x$, $y_2 = e^{-2x}\sin x$; $W = e^{-4x}$; f.s.s. $= \{y_1, y_2\}$

(iii) $m = \pm 2$; $y_1 = e^{2x}$, $y_2 = e^{-2x}$; $W(y_1, y_2)(x) = -4$; f.s.s. $= \{e^{2x}, e^{-2x}\}$

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6. $y_1 = x$, $y_1' = 1$, $y_1'' = 0$. Then $(1-x^2)y_1'' - 2xy_1' + 2y_1 = 0 - 2x + 2x = 0$

Second solution given by $y_2 = y_1 v = y_1 \int \frac{\exp\{-\int p(x)\,dx\}}{y_1^2}\,dx$

Here $p(x) = -\frac{2x}{(1-x^2)}$ so $\int p(x)\,dx = \ln(1-x^2)$ giving $v = \int \frac{dx}{x^2(1-x^2)}$.

Integrate by expanding in partial fractions and obtain

$$y_2 = xv = \tfrac{1}{2}x\ln\left(\tfrac{1+x}{1-x}\right) - 1$$

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