DIFFERENTIAL EQUATIONS
Solutions to Sheet 2

1. Homog. eqⁿ: $y'' + 4y = 0 \Rightarrow y_1 = \cos 2x,\ y_2 = \sin 2x$

$W[y_1, y_2](x) = 2 \Rightarrow \{y_1, y_2\}$ is a f.s.s.

Particular integral: Let $y_p(x) = u_1(x)\cos 2x + u_2(x)\sin 2x$

Using the condition $u_1'\cos 2x + u_2'\sin 2x = 0$

then substituting y_p in d.e. $\Rightarrow -2u_1'\sin 2x + 2u_2'\cos 2x = \sec 2x$

Solving $u_1' = -\frac{1}{2}\tan 2x \Rightarrow u_1(x) = \frac{1}{4}\ln(\cos 2x)$

$u_2' = \frac{1}{2} \Rightarrow u_2(x) = \frac{1}{2}x$

$\therefore y_p(x) = \frac{1}{4}\ln(\cos 2x)\cos 2x + \frac{1}{2}x\sin 2x$

General solution: $y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\cos 2x\ln(\cos 2x) + \frac{1}{2}x\sin 2x$

2. Homog. eqⁿ: $4x^2y'' + 9xy' + y = 0$

Let $y = x^r$. Then $x^r[4r(r-1) + 9r + 1] = 0 \Rightarrow 4r^2 + 5r + 1 = 0 \Rightarrow r = -\frac{1}{4}, -1$

$\therefore y_1 = x^{-\frac{1}{4}},\ y_2 = x^{-1}$

$W[y_1, y_2](x) = -\frac{3}{4}x^{-\frac{9}{4}} \neq 0 \Rightarrow \{x^{-\frac{1}{4}}, x^{-1}\}$ a f.s.s.

Particular integral: Let $y_p(x) = u_1(x)x^{-\frac{1}{4}} + u_2(x)x^{-1}$

Gives $u_1(x) = \frac{4}{9}x^{3/4},\ u_2(x) = -\frac{2}{9}x^{3/2}$

so that $y_p(x) = \frac{2}{9}x^{\frac{1}{2}}$

General solution: $y(x) = c_1x^{-\frac{1}{4}} + c_2x^{-1} + \frac{2}{9}x^{\frac{1}{2}}$

Initial conditions:
$$\left.\begin{aligned} y(1) &= c_1 + c_2 + \tfrac{2}{9} = 1 \\ y'(1) &= -\tfrac{1}{4}c_1 - c_2 + \tfrac{1}{9} = -1 \end{aligned}\right\} \Rightarrow c_1 = -\frac{4}{9},\ c_2 = \frac{11}{9}$$

Solution: $y(x) = -\frac{4}{9}x^{-\frac{1}{4}} + \frac{11}{9}x^{-1} + \frac{2}{9}x^{\frac{1}{2}}$

3. (i) Homog. eqⁿ: $y_1 = e^x,\ y_2 = e^{2x}$ $\quad W[y_1, y_2](x) = e^{3x} \neq 0 \Rightarrow \{y_1, y_2\}$ f.s.s.

I.v.p.: $K(x,t) = \dfrac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W[y_1, y_2](t)} = \dfrac{e^te^{2x} - e^xe^{2t}}{e^{3t}} = e^{2(x-t)} - e^{(x-t)}$

$y(x) = \int_0^x \{e^{2(x-t)} - e^{(x-t)}\}f(t)\,dt$

(ii) Solution: $y(x) = c_1e^x + c_2e^{2x} + \int_0^x \{e^{2(x-t)} - e^{(x-t)}\}f(t)\,dt$

4. Consider the set V and the scalar field $\mathbb{R}$.
We note that the following properties hold for elements of V:

commutativity: $f + g = g + f$ for all $f, g \in V$

associativity: $(f+g)+h = f+(g+h)$ for all $f, g, h \in V$

additive identity: $L[0] = 0 \Rightarrow 0 \in V$ such that $f + 0 = f$ for all $f \in V$

additive inverse: $L[-f] = -L[f] = 0 \Rightarrow$ for every $f \in V$ there exists $(-f) \in V$ such that $(-f) + f = 0$

multiplicative identity: $1 \times f = f$ for all $f \in V$

distributive properties: $a(f+g) = af + ag$, $(a+b)f = af + bf$ for all $f, g \in V$, $a, b \in \mathbb{R}$

Thus V is a vector space over $\mathbb{R}$

Basis: If $\{f, g\}$ is a fundamental set of solutions for $L[\phi] = 0$ then the set of all linear combinations
$af + bg$, for $a, b \in \mathbb{R}$,
contains all the solutions of $L[\phi] = 0$ since it is the general solution.
Hence $\{f, g\}$ spans V. Any set of linearly independent vectors which span V is a basis for V

Note: This theorem extends to higher order equations.

5. P.I.: $y_p(x)$ satisfies the differential equation, by substitution

Homog. eq: $y^{(4)} - y = 0$ Let $y = e^{mx}$. Then $m^4 - 1 = (m^2-1)(m^2+1) = 0$

Thus $m = \pm 1, \pm i$ giving

$y_h(x) = c_1\cosh x + c_2\sinh x + c_3\cos x + c_4\sin x$

General solution: $y(x) = y_h(x) + y_p(x)$
$= c_1\cosh x + c_2\sinh x + c_3\cos x + c_4\sin x + \sin(x^2)$

6. F.s.s.: $L[1] = 0,\ L[e^{-x}] = e^{-x}(-1-2+3) = 0,\ L[e^{3x}] = e^{3x}(27-18-9) = 0$

$W[y_1, y_2, y_3](x) = -12e^{2x} \neq 0 \Rightarrow \{1, e^{-x}, e^{3x}\}$ a f.s.s.

$\{y_0, y_1, y_2\}$: Let $y_i = a_i 1 + b_ie^{-x} + c_ie^{3x}$, $i = 0, 1, 2$

For y_0:
$$\left.\begin{aligned} y_0(0) &= a_0 + b_0 + c_0 = 1 \\ y_0'(0) &= -b_0 + 3c_0 = 0 \\ y_0''(0) &= b_0 + 9c_0 = 0 \end{aligned}\right\} \Rightarrow c_0 = 0,\ b_0 = 0,\ a_0 = 1$$
giving $y_0(x) = 1$

For y_1: $y_1(0) = 0,\ y_1'(0) = 1,\ y_1''(0) = 0$ giving $y_1(x) = \frac{2}{3} - \frac{3}{4}e^{-x} + \frac{1}{12}e^{3x}$

For y_2: $y_2(0) = 0,\ y_2'(0) = 0,\ y_2''(0) = 0$ giving $y_2(x) = -\frac{1}{3} + \frac{1}{4}e^{-x} + \frac{1}{12}e^{3x}$