X3

DIFFERENTIAL EQUATIONS

SOLUTION SHEET 3

1. (i) $(xy')' + \left(x - \frac{p^2}{x}\right)y = 0$

(ii) $(e^{-x^2}y')' + 2e^{-x^2}y = 0$

(iii) $\left(\frac{y'}{x+1}\right)' + \frac{2x}{(x+1)^2}y = 0$

2. $y'' - y = 0 \Rightarrow y_1(x) = \cosh x, \quad y_2(x) = \sinh x; \quad p(x) = 1$

$$W[y_1, y_2](t) = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1$$

$$R(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{p(t)W[y_1,y_2](t)} = \frac{\cosh t \sinh x - \cosh x \sinh t}{1.1} = \sinh(x-t)$$

Solution $\quad y(x) = \int_0^x \sinh(x-t)\, f(t)\,dt$

3. Homog eq: $\quad y'' + y' - 2y = 0 \Rightarrow m^2 + m - 2m = (m+2)(m-1) = 0 \Rightarrow y_1 = e^{-2x}, \; y_2 = e^x$

Wronskian: $\quad W[y_1, y_2](t) = \begin{vmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{vmatrix} = 3e^{-t}$

Self-adjoint form: $\quad p(x) = \exp\{\int 1\,dx\} = e^x$ giving $\frac{d}{dx}(e^x y') - 2e^x y = e^{2x}$

Influence function:
$$R(x,t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{p(t)W[y_1,y_2](t)} = \frac{e^{-2t}e^x - e^{-2x}e^t}{e^t(3e^{-t})} = \frac{1}{3}\left(e^{x-2t} - e^{t-2x}\right)$$

Homog. i.c.s.:
$$\tilde{y}(x) = \int_0^x R(x,t)h(t)\,dt = \int_0^x \frac{1}{3}\left(e^{x-2t} - e^{t-2x}\right)e^{2t}\,dt = \int_0^x \frac{1}{3}\left(e^x - e^{3t-2x}\right)dt = \left[\frac{t}{3}e^x - \frac{1}{9}e^{3t-2x}\right]_0^x = \frac{1}{3}xe^x - \frac{1}{9}e^x + \frac{1}{9}e^{-2x}$$

Nonhomog i.c.s:
$$y(x) = c_1 e^{-2x} + c_2 e^x + \tilde{y}(x)$$
$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 1 \\ y'(0) &= -2c_1 + c_2 = 0 \end{aligned}\right\} \Rightarrow c_1 = \tfrac{1}{3}, \; c_2 = \tfrac{2}{3}$$

Solution:
$$y(x) = \frac{1}{3}e^{-2x} + \frac{2}{3}e^x + \left(\frac{1}{3}xe^x - \frac{1}{9}e^x + \frac{1}{9}e^{-2x}\right) = \frac{1}{9}\left(4e^{-2x} + 5e^x + 3xe^x\right)$$

4. $\lambda = -\mu^2 < 0$: $\quad y'' - \mu^2 y = 0$

g.s. $\quad y = A\cosh\mu x + B\sinh\mu x$

$\quad y' = \mu A\sinh\mu x + \mu B\cosh\mu x$

b.c.s $\quad y'(0) = \mu B = 0 \Rightarrow B = 0$

$\quad y'(L) = \mu A\sinh\mu L = 0 \Rightarrow A = 0$

$\therefore$ no nontrivial solutions

$\lambda = 0$: $\quad y'' = 0$

g.s. $\quad y = Ax + B \Rightarrow y' = A$

b.c.s $\quad y'(0) = A = 0$

$\quad y'(L) = A = 0$

$\therefore A = 0, \; B$ arbitrary

$\therefore \lambda = 0$ is an eigenvalue, $\phi_0(x) = c_0$ the eigenfunction

$\lambda = \mu^2 > 0$: $\quad y'' + \mu^2 y = 0$

g.s. $\quad y = A\cos\mu x + B\sin\mu x$

$\quad y' = -\mu A\sin\mu x + \mu B\cos\mu x$

b.c.s. $\quad y'(0) = \mu B = 0 \Rightarrow B = 0$

$\quad y'(L) = -\mu A\sin\mu L = 0 \Rightarrow \mu L = n\pi$ for all $n \in \mathbb{Z}$

$\therefore \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = c_n\cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$

Hence the problem has eigenvalues $\frac{n^2\pi^2}{L^2}$, $n = 0, 1, 2, \ldots$ e fns $\cos\left(\frac{n\pi x}{L}\right)$

5. $\lambda_n = n^2, \quad \phi_n(x) = c_n\cos nx + d_n\sin nx, \quad n = 0, 1, 2, \ldots$

6. $\lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad \phi_n(x) = c_n e^{-2x}\sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \ldots$

Dr A.C. Bryan 3/01