SOLUTION SHEET 4

Homog. eq.: $y'' = 0 \Rightarrow y = ax + b$, $p(x) = 1$

Green's fn: (a) $G(x,t) = \begin{cases} A(t)x + B(t), & 0 \le x \le t, \\ C(t)x + D(t), & t \le x \le 1 \end{cases}$

(b) $y(0) = 0 \Rightarrow G(0,t) = B = 0$

$y(1) + y'(1) = 0 \Rightarrow G(1,t) + G_x(1,t) = (C+D) + C = 0 \Rightarrow D = -2C$

$\therefore G(x,t) = \begin{cases} A(t)x, & 0 \le x \le t, \\ C(t)(x-2), & t \le x \le 1 \end{cases}$

(c) G continuous at $x=t$: $At = C(t-2)$

(d) $\left.\frac{\partial G}{\partial x}\right|_{x=t+0} - \left.\frac{\partial G}{\partial x}\right|_{x=t-0} = -\frac{1}{1} \Rightarrow C - A = -1$

solve $A = 1 - \frac{1}{2}t$, $C = -\frac{1}{2}t$

$\therefore G(x,t) = \begin{cases} \frac{1}{2}(2-t)x, & 0 \le x \le t, \\ \frac{1}{2}t(2-x), & t \le x \le 1 \end{cases}$

Solution: $y(x) = \int_0^1 G(x,t) f(t)\,dt$

$f(t) = t \Rightarrow y(x) = \int_0^x \{\frac{1}{2}t(2-x)\} t\,dt + \int_x^1 \{\frac{1}{2}(2-t)x\} t\,dt$

$= \frac{1}{2}(2-x)\int_0^x t^2 dt + \frac{1}{2}x \int_x^1 (2t - t^2)\,dt$

$= \frac{1}{3}x - \frac{1}{6}x^3$

Homog. eq.: $y'' + k^2 y = 0$, $y_1 = \cos kx$, $y_2 = \sin kx$, $p(x) = 1$

Uniqueness: $\Delta = \begin{vmatrix} y_1'(0) & y_2'(0) \\ y_1(1) & y_2(1) \end{vmatrix} = \begin{vmatrix} 0 & k \\ \cos k & 0 \end{vmatrix} = -k\cos k$

$\therefore \Delta \ne 0$ if $\cos k \ne 0$, i.e. $k \ne (n+\frac{1}{2})\pi$

Green's fn.: (a) $G(x,t) = \begin{cases} A(t)\cos kx + B(t)\sin kx, & 0 \le x \le t, \\ C(t)\cos kx + D(t)\sin kx, & t \le x \le 1. \end{cases}$

(b) b.c.s $y'(0) = 0 \Rightarrow G_x(0,t) = kB = 0 \Rightarrow B = 0$

$y(1) = 0 \Rightarrow G(1,t) = C\cos k + D\sin k \Rightarrow D = -C\cot k$

$\therefore G(x,t) = \begin{cases} A(t)\cos kx, & 0 \le x \le t \\ \frac{D(t)\sin k(1-x)}{\sin k}, & t \le x \le 1 \end{cases}$

Symmetry: $G(x,t) = \begin{cases} K \sin k(1-t)\cos kx, & 0 \le x \le t \\ K\cos kt \sin k(1-x), & t \le x \le 1 \end{cases}$ (K constant)

(c) G continuous at $x=t$: satisfied

(d) $\frac{\partial G}{\partial x}$ discts at $x=t$: $-K\cos kt\cos k(1-t) + K\sin k(1-t)\sin kt = -\frac{1}{1}$

$\therefore K = \sec k$

$\therefore G(x,t) = \begin{cases} \sec k \sin k(1-t)\cos kx, & 0 \le x \le t, \\ \sec k \cos kt \sin k(1-x), & t \le x \le 1 \end{cases}$

(iii) Solution: $y(x) = \int_0^1 G(x,t).1\,dt$

$= \frac{\sin k(1-x)}{\cos k}\int_0^x \cos kt\,dt + \frac{\cos kx}{\cos k}\int_x^1 \sin k(1-t)\,dt$

$= \frac{\cos kx}{k\cos k} - \frac{1}{k}$

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3. Homog. eq.: $xy'' + y' = 0 \Rightarrow (xy')' = 0 \Rightarrow xy' = A \Rightarrow y = A\ln x + B$

Self-adjoint form: $(xy')' = -f(x) \Rightarrow p(x) = x$

Green's function: (a) $G(x,t) = \begin{cases} A(t)\ln x + B(t), & 1 \le x \le t, \\ C(t)\ln x + D(t), & t \le x \le 2. \end{cases}$

(b) b.c.s $y(1) - y'(1) = 0 \Rightarrow (A\ln 1 + B) - \frac{A}{1} = 0 \Rightarrow B = A$

$y'(2) = 0 \Rightarrow \frac{C}{2} = 0 \Rightarrow C = 0$

$\therefore G(x,t) = \begin{cases} A\ln x + A, & 1 \le x \le t, \\ D, & t \le x \le 2 \end{cases}$

(c) G continuous at $x=t$: $A\ln t + A = D$

(d) $\left.\frac{\partial G}{\partial x}\right|_{x=t+0} - \left.\frac{\partial G}{\partial x}\right|_{x=t-0} = -\frac{1}{p(t)}$: $0 - \frac{A}{t} = -\frac{1}{t}$

$\Rightarrow A = 1$, $D = 1 + \ln t$

$\therefore G(x,t) = \begin{cases} 1 + \ln x, & 1 \le x \le t, \\ 1 + \ln t, & t \le x \le 2. \end{cases}$

Solution: $y(x) = \int_1^2 G(x,t) f(t)\,dt$

$= \int_1^x (1 + \ln t) f(t)\,dt + \int_x^2 (1 + \ln x) f(t)\,dt$

$= \int_1^x (1 + \ln t) f(t)\,dt + (1 + \ln x)\int_x^2 f(t)\,dt$

Verify: Differentiating w.r.t. x gives

$$y'(x) = (1+\ln x) f(x) + \frac{1}{x}\int_x^2 f(t)dt - (1+\ln x) f(x)$$

$$= \frac{1}{x}\int_x^2 f(t)dt$$

$$y''(x) = -\frac{1}{x^2}\int_x^2 f(t)dt + \frac{1}{x}\left(-f(x)\right)$$

Subst. in d.e.: $xy'' + y' = -\frac{1}{x}\int_x^2 f(t)dt - f(x) + \frac{1}{x}\int_x^2 f(t)dt = -f(x)$

Subst in b.c.s: $y(1) - y'(1) = (1+\ln 1)\int_1^2 f(t)dt - \frac{1}{1}\int_1^2 f(t)dt = 0$

$$y'(2) = \frac{1}{2}\int_2^2 f(t)dt = 0$$

$\therefore y(x) = \int_1^2 G(x,t) f(t)dt$ is a solution of the problem

Uniqueness: $y_1(x) = \ln x, \quad y_2(x) = 1$

$$\Delta = \begin{vmatrix} y_1(1) - y_1'(1) & y_2(1) - y_2'(1) \\ y_1'(2) & y_2'(2) \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{2}$$

Since $\Delta \neq 0$, the solution is unique.

4.(i) **Step 1** Obtain appropriate evalues and efns (given here)

$$\lambda_n = n^2\pi^2, \qquad \varphi_n(x) = \sin(n\pi x), \qquad n = 1,2,\ldots$$

Normalisation constant $= \int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 \frac{1}{2}(1-\cos 2n\pi x)\,dx = \frac{1}{2}$

Normalised eigenfunctions: $\hat{\varphi}_n(x) = \sqrt{2}\sin(n\pi x)$

Step 2 Expand r.h.s. of d.e. in terms of the efns.

$$x = \sum_{n=1}^{\infty}\gamma_n\hat{\varphi}_n(x) = \sum_{n=1}^{\infty}\gamma_n\sqrt{2}\sin(n\pi x)$$

$$\gamma_n = \int_0^1 x\sqrt{2}\sin(n\pi x)\,dx = \left[-x\sqrt{2}\frac{\cos(n\pi x)}{n\pi}\right]_0^1 + \int_0^1\frac{\sqrt{2}\cos(n\pi x)}{n\pi}dx = \frac{-(-1)^n\sqrt{2}}{n\pi}$$

Step 3 Let $y = \sum_{n=1}^{\infty} c_n\hat{\varphi}_n(x) = \sum_{n=1}^{\infty} c_n\sqrt{2}\sin(n\pi x)$

Substituting the series into the d.e. gives

$$-\sum_{n=1}^{\infty} n^2\pi^2 c_n\sqrt{2}\sin(n\pi x) + 3\sum_{n=1}^{\infty} c_n\sqrt{2}\sin(n\pi x) = \sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{n\pi}\sqrt{2}\right\}\sqrt{2}\sin(n\pi x)$$

Equating coefficients of $\hat{\varphi}_n(x)$ gives

$$c_n(3 - n^2\pi^2) = \frac{(-1)^n}{n\pi}\sqrt{2}$$

Solution is

$$y(x) = \sum_{n=1}^{\infty}\frac{2(-1)^n\sin(n\pi x)}{n\pi(3-n^2\pi^2)}$$

(unique since $\mu \neq \lambda_n$ for any n)

(ii) **Green's function:** Replace x by $h(x)$ on the r.h.s. of the d.e.

Efn expansion: $h(x) = \sum_{n=1}^{\infty}\gamma_n\hat{\varphi}_n(x)$ where $\gamma_n = \int_0^1 h(t)\sqrt{2}\sin(n\pi t)dt$

Substituting $y = \sum c_n\hat{\varphi}_n(x)$ gives

$$-\sum_{n=1}^{\infty} n^2\pi^2 c_n\hat{\varphi}_n(x) + 3\sum_{n=1}^{\infty} c_n\hat{\varphi}_n(x) = \sum_{n=1}^{\infty}\gamma_n\hat{\varphi}_n(x)$$

$$\therefore c_n = \frac{\gamma_n}{(3-n^2\pi^2)}$$

Solution is

$$y(x) = \sum_{n=1}^{\infty}\frac{\gamma_n}{(3-n^2\pi^2)}\hat{\varphi}_n(x)$$

$$= \sum_{n=1}^{\infty}\left\{\frac{\int_0^1 h(t)\sqrt{2}\sin(n\pi t)dt}{3-n^2\pi^2}\right\}\sqrt{2}\sin(n\pi x)$$

$$= \int_0^1\left\{\sum_{n=1}^{\infty}\frac{2\sin(n\pi t)\sin(n\pi x)}{3-n^2\pi^2}\right\}h(t)\,dt$$

Hence

$$G(x,t) = \sum_{n=1}^{\infty}\frac{2\sin(n\pi t)\sin(n\pi x)}{(3-n^2\pi^2)}$$

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