X3

DIFFERENTIAL EQUATIONS
SOLUTION SHEET 5

1.(i) $\underline{y}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\underline{y}$ $\quad\therefore A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

evalues of A: $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)-6 = \lambda^2-3\lambda-4 = 0$

$\therefore \lambda = 4, -1$

evectors: $(A-\lambda I)\underline{u} = \underline{0}$: $\lambda = 4 \Rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2u_1 - 3u_2 = 0 \Rightarrow \underline{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda = -1 \Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_1 + u_2 = 0 \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

general solution: $\underline{y} = c_1 e^{4x}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) $\underline{y}' = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}\underline{y}$ $\quad\therefore A = \begin{pmatrix} -2 & 2 \\ -1 & 0 \end{pmatrix}$

evalues of A: $\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = \lambda^2+2\lambda+2 = 0$ $\quad\therefore \lambda = -1 \pm i$

evectors: $(A-\lambda I)\underline{u} = \underline{0}$: $\lambda = -1+i \Rightarrow \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -(1+i)u_1 + 2u_2 = 0 \\ -u_1 + (1-i)u_2 = 0 \end{matrix} \Rightarrow \underline{u} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

$\therefore e^{\lambda x}\underline{u} = e^{(-1+i)x}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} = e^{-x}(\cos x + i\sin x)\begin{pmatrix} 1-i \\ 1 \end{pmatrix} = e^{-x}\begin{pmatrix} c+s \\ c \end{pmatrix} + ie^{-x}\begin{pmatrix} s-c \\ s \end{pmatrix}$

general solution: $\underline{y} = c_1 e^{-x}\begin{pmatrix} \cos x + \sin x \\ \cos x \end{pmatrix} + c_2 e^{-x}\begin{pmatrix} \sin x - \cos x \\ \sin x \end{pmatrix}$

2.(i) $\underline{y}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\underline{y}$ $\quad\therefore A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, evalues 2, 1; evectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

transⁿ: $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow J = P^{-1}AP = \frac{1}{(-1)}\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

system $\underline{y}' = A\underline{y}$ with $\underline{y} = P\underline{z} \Rightarrow P\underline{z}' = AP\underline{z} \Rightarrow \underline{z}' = P^{-1}AP\underline{z} = J\underline{z}$

$\underline{z}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\underline{z} \Rightarrow \begin{matrix} z_1' = 2z_1 \\ z_2' = z_2 \end{matrix} \Rightarrow \begin{matrix} z_1 = c_1 e^{2x} \\ z_2 = c_2 e^{x} \end{matrix}$

initial condition $\underline{z}(0) = P^{-1}\underline{y}(0) = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = -2 \\ c_2 = 3 \end{matrix}$

solution $\underline{y} = P\underline{z} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -2e^{2x} \\ 3e^{x} \end{pmatrix} = \begin{pmatrix} -2e^{2x} + 3e^{x} \\ -4e^{2x} + 3e^{x} \end{pmatrix}$

(ii) $\underline{y}' = \begin{pmatrix} 9 & 4 \\ -9 & -3 \end{pmatrix}\underline{y}$ $\quad\therefore A = \begin{pmatrix} 9 & 4 \\ -9 & -3 \end{pmatrix}$, evalues 3, 3; evector $\underline{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

generalised evector: $(A-3I)\underline{v} = \underline{u} \Rightarrow \begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \Rightarrow 3v_1 + 2v_2 = 1 \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

transⁿ: $P = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \Rightarrow J = P^{-1}AP = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

system: $\underline{z}' = J\underline{z} \Rightarrow \underline{z}' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\underline{z} \Rightarrow \begin{matrix} z_1' = 3z_1 + z_2 \\ z_2' = 3z_2 \end{matrix}$

solve for z_2: $z_2 = c_2 e^{3x}$

" " z_1: $z_1' - 3z_1 = c_2 e^{3x}$ $\quad\therefore (e^{-3x}z_1)' = c_2$ $\quad\therefore e^{-3x}z_1 = c_2 x + c_1$ $\quad\therefore z_1 = (c_1 + c_2$

solution: $\underline{y} = P\underline{z} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} (c_1 + c_2 x)e^{3x} \\ c_2 e^{3x} \end{pmatrix} = \begin{pmatrix} 2c_1 e^{3x} + c_2 e^{3x} + 2c_2 x e^{3x} \\ -3c_1 e^{3x} - c_2 e^{3x} - 3c_2 x e^{3x} \end{pmatrix}$

initial conditions: $\underline{y}(0) = \begin{pmatrix} 2c_1 + c_2 \\ -3c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow c_1 = 0, \quad c_2 = 1$

$\therefore \underline{y}(x) = \begin{pmatrix} (1+2x)e^{3x} \\ -(1+3x)e^{3x} \end{pmatrix}$

3. $\underline{y}' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\underline{y} + \begin{pmatrix} -7e^{-2x} \\ 0 \end{pmatrix}$

evalues/evectors: $A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \Rightarrow \lambda = 5, \ \underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \ \lambda = -2, \ \underline{u} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

coord transⁿ: $P = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \Rightarrow J = P^{-1}AP = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}, \quad P^{-1} = \frac{1}{(-7)}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix}$

canonical system: $\underline{y}' = A\underline{y} + \underline{h}$ with $\underline{y} = P\underline{z} \Rightarrow P\underline{z}' = AP\underline{z} + \underline{h} \Rightarrow \underline{z}' = P^{-1}AP\underline{z} + P^{-1}\underline{h}$

$\therefore \underline{z}' = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}\underline{z} - \frac{1}{7}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -7e^{-2x} \\ 0 \end{pmatrix}$

$\therefore z_1' = 5z_1 - 3e^{-2x}$

$z_2' = -2z_2 - e^{-2x}$

solve for z_1: $z_1' - 5z_1 = -3e^{-2x} \Rightarrow (e^{-5x}z_1)' = -3e^{-7x}$

$\Rightarrow e^{-5x}z_1 = \frac{3}{7}e^{-7x} + c_1$

$\Rightarrow z_1 = \frac{3}{7}e^{-2x} + c_1 e^{5x}$

solve for z_2: $z_2' + 2z_2 = -e^{-2x} \Rightarrow (e^{2x}z_2)' = -1$

$\Rightarrow e^{2x}z_2 = -x + c_2$

$\Rightarrow z_2 = -xe^{-2x} + c_2 e^{-2x}$

solution: $\underline{y} = P\underline{z} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} c_1 e^{5x} + \frac{3}{7}e^{-2x} \\ c_2 e^{-2x} - xe^{-2x} \end{pmatrix}$

$\therefore y_1(x) = 2c_1 e^{5x} + c_2 e^{-2x} + \left(\frac{6}{7} - x\right)e^{-2x}$

$y_2(x) = c_1 e^{5x} - 3c_2 e^{-2x} + \left(\frac{3}{7} + 3x\right)e^{-2x}$

4.(a) From Question 1 $\quad \underline{y}_1(x) = e^{4x}\begin{pmatrix}3\\2\end{pmatrix}, \quad \underline{y}_2(x) = e^{-x}\begin{pmatrix}1\\-1\end{pmatrix}$

Let $\quad M(x) = (\underline{y}_1 \,\vdots\, \underline{y}_2) = \begin{pmatrix}3e^{4x} & e^{-x}\\ 2e^{4x} & -e^{-x}\end{pmatrix}$

$$\det M(x) = -5e^{3x}$$

Since $\det M(x) \neq 0$, then $M(x)$ is a fundamental matrix

(b) Transition matrix:

$$M(x, x_0) = M(x)\,[M(x_0)]^{-1}$$

$$[M(x)]^{-1} = \frac{1}{(-5e^{3x})}\begin{pmatrix}-e^{-x} & -e^{-x}\\ -2e^{4x} & 3e^{4x}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}e^{-4x} & e^{-4x}\\ 2e^{x} & -3e^{x}\end{pmatrix}$$

$$M(x, x_0) = \frac{1}{5}\begin{pmatrix}3e^{4x} & e^{-x}\\ 2e^{4x} & -e^{-x}\end{pmatrix}\begin{pmatrix}e^{-4x_0} & e^{-4x_0}\\ 2e^{x_0} & -3e^{x_0}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{3}{5}e^{4(x-x_0)} + \frac{2}{5}e^{-(x-x_0)} & \frac{3}{5}e^{4(x-x_0)} - \frac{3}{5}e^{-(x-x_0)}\\ \frac{2}{5}e^{4(x-x_0)} - \frac{2}{5}e^{-(x-x_0)} & \frac{2}{5}e^{4(x-x_0)} + \frac{3}{5}e^{-(x-x_0)}\end{pmatrix}$$

(c) solution: $\quad \underline{y}(x) = M(x, 0)\,\underline{y}(0)$

$$= \begin{pmatrix}\frac{3}{5}e^{4x} + \frac{2}{5}e^{-x} & \frac{3}{5}e^{4x} - \frac{3}{5}e^{-x}\\ \frac{2}{5}e^{4x} - \frac{2}{5}e^{-x} & \frac{2}{5}e^{4x} + \frac{3}{5}e^{-x}\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}$$

$$= \begin{pmatrix}\frac{3}{5}e^{4x} - \frac{8}{5}e^{-x}\\ \frac{2}{5}e^{4x} + \frac{8}{5}e^{-x}\end{pmatrix}$$

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5. Transition matrix:

$$M'(x, x_0) = \begin{pmatrix}-x\sin\frac{1}{2}(x^2 - x_0^2) & x\cos\frac{1}{2}(x^2 - x_0^2)\\ -x\cos\frac{1}{2}(x^2 - x_0^2) & -x\sin\frac{1}{2}(x - x_0^2)\end{pmatrix}$$

$$= \begin{pmatrix}0 & x\\ -x & 0\end{pmatrix}\begin{pmatrix}\cos\frac{1}{2}(x^2 - x_0^2) & \sin\frac{1}{2}(x^2 - x_0^2)\\ -\sin\frac{1}{2}(x^2 - x_0^2) & \cos\frac{1}{2}(x^2 - x_0^2)\end{pmatrix}$$

$$= A(x)\,M(x, x_0)$$

$$\det M(x, x_0) = \cos^2\tfrac{1}{2}(x^2 - x_0^2) + \sin^2\tfrac{1}{2}(x - x_0^2) = 1$$

$\therefore M(x, x_0)$ is a fundamental matrix

Since $M(x_0, x_0) = I$ then $M(x, x_0)$ is a transition matrix

Solution $\quad \underline{y}(x) = M(x,0)\,\underline{y}(0) = \begin{pmatrix}\cos\frac{1}{2}x^2 & \sin\frac{1}{2}x^2\\ -\sin\frac{1}{2}x^2 & \cos\frac{1}{2}x^2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}2\cos(\frac{1}{2}x^2) - \sin(\frac{1}{2}x^2)\\ -2\sin(\frac{1}{2}x^2) - \cos(\frac{1}{2}x^2)\end{pmatrix}$

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6. Let $\quad \underline{y}(x_1) = M(x_1, x_0)\,\underline{y}(x_0)$

and $\quad \underline{y}(x_2) = M(x_2, x_1)\,\underline{y}(x_1)$

Then $\quad \underline{y}(x_2) = M(x_2, x_1)\,M(x_1, x_0)\,\underline{y}(x_0) \quad$ and $\quad \underline{y}(x_2) = M(x_2, x_0)\,\underline{y}(x_0)$

Hence $\quad M(x_2, x_0) = M(x_2, x_1)\,M(x_1, x_0)$

Using $M(x, x_0)$ from Question 5

$$M(x_2, x_1)\,M(x_1, x_0) = \begin{pmatrix}\cos\frac{1}{2}(x_2^2 - x_1^2) & \sin\frac{1}{2}(x_2^2 - x_1^2)\\ -\sin\frac{1}{2}(x_2^2 - x_1^2) & \cos\frac{1}{2}(x_2^2 - x_1^2)\end{pmatrix}\begin{pmatrix}\cos\frac{1}{2}(x_1^2 - x_0^2) & \sin\frac{1}{2}(x_1^2 - x_0^2)\\ -\sin\frac{1}{2}(x_1^2 - x_0^2) & \cos\frac{1}{2}(x_1^2 - x_0^2)\end{pmatrix}$$

$$= \begin{pmatrix}\cos\left[\frac{1}{2}(x_2^2 - x_1^2) + \frac{1}{2}(x_1^2 - x_0^2)\right] & \sin\left[\frac{1}{2}(x_2^2 - x_1^2) + \frac{1}{2}(x_1^2 - x_0^2)\right]\\ -\sin\left[\frac{1}{2}(x_2^2 - x_1^2) + \frac{1}{2}(x_1^2 - x_0^2)\right] & \cos\left[\frac{1}{2}(x_2^2 - x_1^2) + \frac{1}{2}(x_1^2 - x_0^2)\right]\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{1}{2}(x_2^2 - x_0^2) & \sin\frac{1}{2}(x_2^2 - x_0^2)\\ -\sin\frac{1}{2}(x_2^2 - x_0^2) & \cos\frac{1}{2}(x_2^2 - x_0^2)\end{pmatrix}$$

$$= M(x_2, x_0)$$

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