DISCRETE MATHEMATICS, EXERCISES SHEET 5

(1) The purpose of this exercise is to show that the Perron-Frobenius theorem, as stated in the lectures for positive stochastic matrices, holds for positivizable stochastic matrices. So let M be a positivizable stochastic matrix. By definition M^r is a positive matrix for some $r \geq 1$.
   (a) Show that M^r is a stochastic matrix. (Thus we may assume that the Perron-Frobenius therem holds for M^r, as proved in the lectures.)
   (b) Show that an invariant measure of M is also an invariant measure of M^r. Hence deduce that M possess an unique invariant measure π, and that π has positive components.
   (c) Show that the sequence $v(0), v(0)M, v(0)M^2, \ldots$ converges to π, for any initial probability distribution $v(0)$. (Hint: split the sequence into subsequences and use the known result for M^r.)
   (d) Use the previous part to show that the sequence of matrices $M, M^2, M^3, \ldots$ converges to the matrix all of whose rows are π. Conclude that that 1 is the only eigenvalue of M of modulus 1, and that its multiplicity is 1.

(2) Let

$$M = \begin{pmatrix} 7 & 3 \\ 6 & 0 \end{pmatrix}.$$

   (a) Show that the spectral radius of M is 9.
   (b) Find a left eigenvector of M with eigenvalue 9.
   (c) Find a right eigenvector of M with eigenvalue 9.
   (d) Use the Perron-Frobenius Theorem to show that

$$\lim_{n\to\infty} \left(\frac{1}{9}M\right)^n$$

   exists and calculate its value.