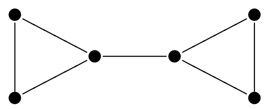
DISCRETE MATHEMATICS, EXERCISES SHEET 8

(1) For each of the polynomials $f(t)$ listed below, find a graph G such that $f(t) = P(G,t)$, or else prove that no such G exists.
   (a) $f(t) = t^2$;
   (b) $f(t) = t^2 - t$;
   (c) $f(t) = t^3 - 4t^2 + 3t$;
   (d) $f(t) = t^4 - 2t^3 + t^2$.

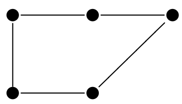
(2) Let G be the graph



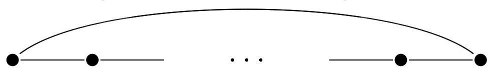
   (a) Find the chromatic number $\chi(G)$ of G and exhibit a colouring using $\chi(G)$ colours.
   (b) Calculate the chromatic polynomial $P(G,t)$ of G.
   (c) Show that $P(G,N)$ is divisible by 8 for any positive integer N.
   (d) Give another explanation for (c), without using explicit knowledge of $P(G,t)$, by considering the symmetries of G.

(3) (a) Let L_n be the graph

with n vertices connected by $n-1$ edges to form a line. Prove that $P(L_n,t) = t(t-1)^{n-1}$.
   (b) Let G be the graph



Show that $P(G,t) = (t-1)^5 - (t-1)$. (Hint: use the contraction-deletion together with part (a)).
   (c) We can generalise the result of part (b). Let C_n be the graph with n vertices obtained from L_n by adding an additional edge connecting the leftmost and rightmost vertices:



(Note that the graph G of part (b) is isomorphic to C_5.) Prove by induction on n that $P(C_n,t) = (t-1)^n + (-1)^n(t-1)$, for $n \geq 1$.