DISCRETE MATHEMATICS, SOLUTIONS SHEET 2

(1) (a) As we saw already from Exercise Sheet 1, Problem 2a, there are 6 possibilities, two of which are

$$M_{ABC}(G) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad M_{BAC}(G) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

The ordering of vertices for the second is obtained from the first via the permutation (12). The corresponding permutation matrix

$$U_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is its own inverse, and we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

(b) Let us take $W = M_{ABC}(G)$. By dividing the entries of the second and third rows by 2, we get the stochastic matrix

$$\overline{W} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

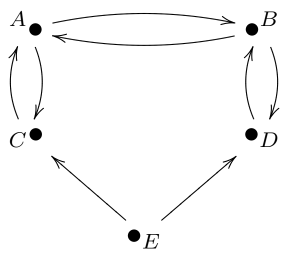
In Problem 4 of Sheet 1, we found that 36 'random surfers' will in the long run converge to a distribution of 12, 16 and 8 surfers at pages A, B and C, respectively. This translates into a probability distribution row vector $v = (1/3, 4/9, 2/9)$, and that this is indeed an invariant measure: $v\overline{W} = v$.

(c) The modified stochastic matrix is

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \end{pmatrix}.$$

There are two ways of getting from B to C in three steps: $B \to A \to B \to C$ and $B \to C \to B \to C$. So the desired probability is $(1/2)(1)(1/2) + (1/2)(2/3)(1/2) = 5/12$. Alternatively, it is the $(2,3)$-entry of the matrix Q^3.

(2) One possibility is given by the model represented by the following digraph G:



There is a symmetry of the graph interchanging A with B, and C with D (and leaving E fixed, which is expressed as an equality of adjacency matrices $M_{ABCDE}(G) = M_{BADCE}G$. So A and B should be ranked equally, and so should C and D. Now E has no inlinks, so in the long run it has no traffic. Finally the only outlink from C goes to A, whereas A also receives traffic from B; thus A should be ranked above C, provided that there are some surfers at B in the long run. And indedd this is so, for if there are none at B in the long run, then there would also be none at D since there is a link from D to B. Then by symmetry there would be no surfers at any page in the long run, which is impossible. A similar argument shows that there are some surfers at C (and D) in the long run, so we arrive at the ranking $A = B > C = D > E$. (Though not required to answer the question, the invariant measure turns out to be $A : 1/3, B : 1/3, C : 1/6, D : 1/6, E : 0$.)

(3) A possible adjacency matrix is

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is already row stochastic, so $\overline{W} = W$. Now (x, x, y, y) is a left eigenvector of $\overline{W}$ with eigenvalue 1, for any complex numbers x and y. Given any real number x such that $0 \leq x \leq \frac{1}{2}$, the vector $(x, x, \frac{1}{2} - x, \frac{1}{2} - x)$ is an invariant measure. So an invariant measure exists but is not unique.

(4) Let w be right eigenvector of M with eigenvalue λ; so w is a nonzero column vector such that $Mw = \lambda w$.

(a) We have $M^2(w) = MM(w) = M(\lambda w) = \lambda M(w) = \lambda^2 w$. Then $M^3(w) = M(M^2(w)) = M(\lambda^2 w) = \lambda^2 M(w) = \lambda^3 w$. By induction on w we obtain $M^k(w) = \lambda^k w$. So w is a right eigenvector of M^k with eigenvalue λ^k; in particular λ^k is an eigenvalue of M^k.

(b) We have $(tM)w = t(M(w)) = t\lambda(w)$. So w is a right eigenvector of tM with eigenvalue $t\lambda$.