DISCRETE MATHEMATICS, SOLUTIONS SHEET 3

(1) Let $v = (v_1, \ldots, v_n)$ and $v' = (v'_1, \ldots, v'_n)$ be two different invariant measures of M. We have $vM = v$, $v'M = v'$, $0 \leq v_i \leq 1$, $0 \leq v'_i \leq 1$, $\sum_{i=1}^n v_i = 1$ and $\sum_{i=1}^n v'_i = 1$. The last two equations imply that v and v' are linearly independent, as they are different. So we can define (infinitely many) distinct vectors $v^\mu = \mu v + (1-\mu)v'$, where $0 \leq \mu \leq 1$.

Then $v^\mu M = \mu vM + (1-\mu)v'M = \mu v + (1-\mu)v' = v^\mu$. Moreover $0 \leq \mu v_i + (1-\mu)v'_i = v_i^\mu \leq \mu + (1-\mu) = 1$ and $\sum_{i=1}^n v_i^\mu = \mu \sum_{i=1}^n v_i + (1-\mu)\sum_{i=1}^n v'_i = \mu + (1-\mu) = 1$.

(2) The characteristic polynomial has the form $(x-2)^s(x+1)^t$, where $s + t = 4$, $s \geq 1$ and $t \geq 1$. So there are three possibilities: $(x-2)(x+1)^3$, $(x-2)^2(x+1)^2$ and $(x-2)^3(x+1)$. The corresponding possibilities for the Jordan form are

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(3) In other words, what is number of (nonsimilar) Jordan forms with characteristic polynomial $(x-2)^2(x-1)^2(x-4)^5$? Well, for the eigenvalue 2 there is either 1 block of size 2 or 2 blocks of size 1; these two possibilities correspond to the partitions $2 = 2$ and $2 = 1 + 1$. For the eigenvalue 1 we have the same story. Finally for the eigenvalue 4 there are 7 possibilities: $5 = 5$, $5 = 4 + 1$, $5 = 3 + 2$, $5 = 3 + 1 + 1$, $5 = 2 + 2 + 1$, $5 = 2 + 1 + 1 + 1$ and $5 = 1 + 1 + 1 + 1 + 1$.

So the total number of possibilities is $2 \cdot 2 \cdot 7 = 28$.

(4) We have $P_M(x) = (3-x)(-1-x) - (4)(-1) = x^2 - 2x + 1 = (x-1)^2$, so M has eigenvalue 1 with multiplicity 2.

If $u = \begin{pmatrix} a \\ b \end{pmatrix}$ is a right eigenvector with eigenvalue 1 then $3a - b = a$ and $4a - b = b$, and so, up to scaling, the only possibility for u is $u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. So M does not have two linearly independent eigenvectors and hence is not diagonaliable. So its Jordan form must be $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. To find the similarity transformation we put $N = M - \lambda I =$

$M-(1)I = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ and look for a solution u_1 to $Nu_2 = u_1$. One solution is $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We deduce that $M = UJU^{-1}$, where $U = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, the matrix whose columns are u_1 and u_2. We have $U^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$. So

$$M^k = UJ^kU^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2k+1 & -k \\ 4k & 1-2k \end{pmatrix}.$$

(5) Let $U = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$. Then U is an invertible matrix with inverse $U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & y^{-1} \end{pmatrix}$, and $U \begin{pmatrix} \lambda & y \\ 0 & \lambda \end{pmatrix} U^{-1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

(6) (a) The first four powers M, M^2, M^3, M^4 of $M = J_4(\lambda)$ are

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda^2 & 2\lambda & 1 & 0 \\ 0 & \lambda^2 & 2\lambda & 1 \\ 0 & 0 & \lambda^2 & 2\lambda \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}, \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda & 1 \\ 0 & \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}, \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 & 4\lambda \\ 0 & \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & 0 & \lambda^4 \end{pmatrix}.$$

(b) The matrix $J_n(\lambda)^k$ is upper triangular, i.e. all entries below the main diagonal are zero. Its diagonal entries are all λ^k. The entries on the diagonal just above the main diagonal are all $k\lambda^{k-1}$. In general, the entries in the $m-th$ diagonal above the main diagonal are $\binom{k}{m}\lambda^{k-m}$ (if $m > k$ the entries are 0).

(c) In light of the previous part, it suffices to show that $\lim_{k\to\infty} \binom{k}{m}\lambda^{k-m} = 0$, for any m. But $|\binom{k}{m}\lambda^{k-m}| \leq (k^m|\lambda|^k)|\lambda|^{-m}$, and it is a standard fact from calculus that $\lim_{k\to\infty} k^m|\lambda|^k = 0$, as $|\lambda| < 1$.