DISCRETE MATHEMATICS, SOLUTIONS SHEET 4

(1) Recall that a $n \times n$ matrix M is row stochastic if each for each entry is nonnegative, and the sum of entries in each row is 1. In symbols these conditions are $M_{ij} \geq 0$ for all i, j, and $\sum_{j=1}^{n} M_{ij} = 1$ for all i.

(a) The (i, j)-entry of the product MN is $(MN)_{ij} = \sum_{k=1}^{n} M_{ik} N_{kj}$. Since sums and products of nonnegative numbers are nonnegative, we have $(MN)_{ij} \geq 0$. And for each i we have $\sum_{j=1}^{n} (MN)_{ij} = \sum_{j=1}^{n} \sum_{k=1}^{n} M_{ik} N_{kj} = \sum_{k=1}^{n} \left(M_{ik} \sum_{j=1}^{n} N_{kj} \right) = \sum_{k=1}^{n} M_{ik} = 1$.

(b) We have $(pM + (1-p)N)_{ij} = pM_{ij} + (1-p)N_{ij} \geq 0$ for any i, j. And for all i, we get $\sum_{j=1}^{n} (pM + (1-p)N)_{ij} = p \sum_{j=1}^{n} M_{ij} + (1-p) \sum_{j=1}^{n} N_{ij} = p + (1-p) = 1$.

(2) (a) Recall that

$$\overline{W} = W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

So the adjusted stochastic matrix is

$$\begin{aligned} W_p &= p\overline{W} + (1-p)F = p \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + (1-p) \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \\ &= \begin{pmatrix} (1-p)/3 & (1+2p)/3 & (1-p)/3 \\ (1-p)/3 & (1-p)/3 & (1+2p)/3 \\ (1+2p)/3 & (1-p)/3 & (1-p)/3 \end{pmatrix} \end{aligned}$$

By symmetry all components of the invariant measure for W_p should be equal. And indeed we can check that for $v = \begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}$ we have $vW_p = v$. Note that the answer doesn't depend on the parameter p. In particular, whatever p we use, the three pages in the associated web model are ranked equally.

(b) Recall that

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and thus

$$\overline{W} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

Hence the adjusted stochastic matrix is

$$\begin{aligned} W_p &= p\overline{W} + (1-p)F = p\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} + (1-p)\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 2-2p & 2+4p & 2-2p \\ 2+p & 2-2p & 2+p \\ 2+p & 2+p & 2-2p \end{pmatrix} \end{aligned}$$

Now suppose (x, y, z) is an invariant measure for W_p. Then $(x, y, z)(W_p - I) = 0$, or equivalently

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -4-2p & 2+4p & 2-2p \\ 2+p & -4-2p & 2+p \\ 2+p & 2+p & -4-2p \end{pmatrix} = 0.$$

Imposing the stochastic condition $x + y + z = 1$ and solving for x, y, z we obtain $v = (\frac{1}{3}, \frac{2+2p}{6+3p}, \frac{2}{6+3p})$. Unlike part (a) the answer does depend on p. As expected, when $p = 1$ we recover the invariant measure $v = (1/3, 4/9, 2/9)$ that we worked out in Exercise Sheet #1, and when $p = 0$ we have $v = (1/3, 1/3, 1/3)$. Note that $2 \leq 2 + 2p \leq 2 + 2p$ since $0 \leq p \leq 1$, with equalities if and only if $p = 0$. So we always have $z < y < x$, unless $p = 0$ in which case $z = y = x$.