DISCRETE MATHEMATICS, SOLUTIONS SHEET 5

(1) The purpose of this exercise is to show that the Perron-Frobenius theorem, as stated in the lectures for positive stochastic matrices, holds for positivizable stochastic matrices. So let M be a positivizable stochastic matrix. By definition M^r is a positive matrix for some $r \geq 1$.

(a) In an earlier exercise we showed that the product of two (row) stochastic matrices is stochastic. By induction the product of any number of stochastic matrices is stochastic, and in particular any power of a stochastic matrix is stochastic.

(b) Suppose π is an invariant measure of M. Then $\pi M^r = (\pi M)M^{r-1} = \pi M^{r-1} = \ldots = \pi$, so π is also an invariant measure of M^r. Since the Perron-Frobenius theorem is assumed to hold for M^r we conclude that π has positive components and is unique.

(c) Consider the sequences

$$\begin{aligned}
&v(0), v(0)M^r, v(0)M^{2r}, \ldots \\
&v(0)M, v(0)M^{r+1}, v(0)M^{2r+1}, \ldots \\
&\vdots \\
&v(0)M^{r-1}, v(0)M^{2r-1}, v(0)M^{3r-1}, \ldots
\end{aligned}$$

By the Perron-Frobenius theorem for applied to M^r, each of these sequences converges to π. It follows that the sequence $v(0), v(0)M, v(0)M^2, \ldots$ also converges to π.

(d) Consider the sequence $I, IM = M, IM^2 = M^2, \ldots$, where I is the $n \times n$ identity matrix. If we just look at the i-th row of the matrices, we get the sequence $v_i, v_iM, v_iM^2, \ldots$, where $v_i = (0, \ldots 0, 1, 0, \ldots, 0)$, with 1 in the i-th spot. By the previous part, each of these sequence converges to π. Thus we deduce that the sequence $M, M^2, M^3, \ldots$ converges to the matrix all of whose rows are π. That matrix has rank 1, so we conclude, just as in the proof of the Perron-Frobenius theorem given in the lectures, that 1 is the only eigenvalue of M of modulus 1, and that its multiplicity is 1.

(2) (a) The characteristic polynomial of M is $P_M(x) = (7-x)(-x) - 3.6 = (x-9)(x+2)$. So the eigenvalues of M are 9 and -2 and its spectral radius is $\max\{|9|, |-2|\} = 9$.

(b) $\pi_L = \begin{pmatrix} 3 & 1 \end{pmatrix}$ is a left eigenvector of M with eigenvalue 9.

(c) $\pi_R = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is a right eigenvector of M with eigenvalue 9.

(d) Even though M is not a positive matrix, it is positivizable since M^2 is positive. So the Perron-Frobenius theorem applies and

$$\lim_{n\to\infty} \left(\frac{1}{9}M\right)^n = \frac{\pi_R \pi_L}{\pi_L \pi_R} = \frac{1}{11}\begin{pmatrix} 9 & 3 \\ 6 & 2 \end{pmatrix}.$$