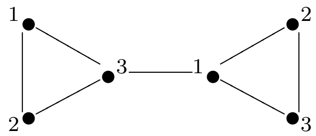
DISCRETE MATHEMATICS, SOLUTIONS SHEET 8

(1) (a) Take G to be a graph with two vertices and no edges. The number of ways to colour G with t colours is t^2 as both vertices can be coloured by any of the t colours.
(b) Let G be a graph with two vertices and one edge connecting them. Once we choose one of t colours for one vertex, any of the remaining $t-1$ can be used to colour the other, so $P(G,t) = t(t-1) = t^2 - t$.
(c) We have $f(2) = -2 < 0$. On the other hand $P(G,2) \geq 0$ for any graph G. So no appropriate graph G exists.
(d) We can write $f(t) = (t^2 - t)^2$. So we may take G to be the disjoint union of two copies of the graph used to answer part (b): G has 4 vertices connected in pairs by two edges.

(2) (a) Here is a colouring with 3 colours:

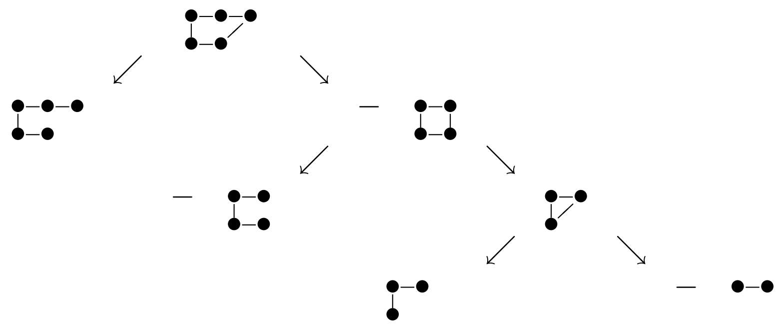


Since there is clearly no colouring with 2 colours, we have $\chi(G) = 2$.
(b) Suppose I am colouring G with t colours. I start at the upper left vertex, colouring it with any one of the t colours. The vertex below can then be coloured by any of the remaining $t-1$ colours. The two vertices already coloured are both connected to a third, for which there are now $t-2$ choices. Moving to the vertex to the immediate right, there are $t-1$ choices, as only the colour just used has to be avoided. Finally, reasoning in the same way, there are $t-1$ and $t-2$ choices for the remaining two vertices. So we have $P(G,t) = t(t-1)^3(t-2)^2$.
(c) If N is even then so is $N-2$, and thus $P(G,N) = N(N-1)^3(N-2)^2$ is divisible by 8. On the other hand if N is odd, then $N-1$ is even and $P(G,N)$ is still divisible by 8.
(d) There is a symmetry of G given by reflection about a horizontal axis. There are also a symmetry interchanging the leftmost two vertices, and one interchanging the rightmost two vertices. By composing these symmetries we obtain 8 distinct symmetries (including the identity symmetry). We define an equivalence relation on the set of colourings of G by N colours, where two colourings are equivalent if one can be obtained from the other by applying a symmetry of G. Each equivalence class has 8 colourings, so the total number of colourings is divisible by 8.

(3) (a) Let us colour the vertices of L_n from left to right. For the leftmost we may choose any of t colours. The next, adjacent, vertex must be a different colour, so there are $t-1$ choices. The third vertex is connected by an edge to the second one, but not the first, so again we have $t-1$ choices. As we move to the right, each vertex is connected with exactly one of the vertices already coloured, so again there are $t-1$ choices. The number of colourings of the whole graph is the product of the

number of colourings we have worked out for the individual vertices, so we obtain $t(t-1)^{n-1}$ as desired.

(b) We illustrate steps in the contraction-deletion algorithm applied to G:



We thus have

$$\begin{aligned} P(G,t) &= P(L_5,t) - P(L_4,t) + P(L_3,t) - P(L_2,t) \\ &= t(t-1)^4 - t(t-1)^3 + t(t-1)^2 - t(t-1) \\ &= [(t-1)+1][(t-1)^4 - (t-1)^3 + (t-1)^2 - (t-1)] \\ &= (t-1)^5 - (t-1). \end{aligned}$$

(c) Since C_1 is a graph with one vertex and one loop, it has no colourings, so $P(C_1,t) = 0$. On the other hand $(t-1)^1 + (-1)^1(t-1) = 0$, so the theorem is true for $n = 1$. Now let $n > 1$, and let e be an edge of C_n. Then by the contraction-deletion theorem, $P(C_n,t) = P(C_n - e, t) - P(C_n/e, t)$. Here $C_n - e$ is obtained by deleting e and thus arriving at a graph isomorphic to L_n, while C_n/e is the graph obtained by contracting e, which is isomorphic to C_{n-1}. Thus $P(C_n,t) = P(L_n,t) - P(C_{n-1},t)$ So by part (a) and by induction, we have $P(C_n,t) = t(t-1)^{n-1} - ((t-1)^{n-1} + (-1)^{n-1}(t-1)) = (t-1)^n + (-1)^n(t-1)$, as desired.