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THE GAME-THEORETICAL MODELLING OF A DYNAMICALLY EVOLVING NETWORK: REVISITING THE TARGET SEQUENCE 111

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Abstract. In previous work we considered a model of a dynamically evolving network of interactions between a group of individuals, where each individual has an optimum level of social engagement with other group members. A randomly selected individual will form or break a link to obtain the required number of contacts. These interactions were formulated as a graph realisation problem. This short paper considers a game-theoretical version of the model, where individuals strategically choose the specific link to form or break. This game is known from previous work to be very complex for all but almost trivial cases, with the exception of an example with three players considered by Broom and Cannings. We revisit this example and show that even this is more complex than previously thought. In this paper, we find a general expression for the payoff functions for all possible strategy combinations. In addition to the three Nash equilibria previously found, we find a set of six more. The considerations of all possibilities proves to be infeasible, leaving the possibility of more solutions open.

1. Introduction.

1.1. A dynamic network population model.

In this paper, we consider a group of individuals, with no predefined graph structure connecting them. Broom and Cannings [2] introduced the dynamic process and investigated its properties on the minimal set of graphs, the graphs for which no mutually beneficial change in the set of links is possible. They considered the properties of the minimal set in [3] and in [4] they further developed the work on the minimal set, and introduced the game-theoretical version of the model that we consider in this paper. We follow an evolutionary process on a set of fixed vertices $v = \{v_1, v_2, ..., v_n\}$ and a set of dynamically evolving edges $X_\tau = x_{ij}; x_{ij} = x_{ji} = 1$ if there is a link between nodes v_i and v_j and $x_{ij} = x_{ji} = 0$ otherwise. This process takes place on a set of simple graphs $\mathbf{G}(v, X)$ which will continue evolving until a minimal graph $G(v, X)$ can be reached.

At each step, an individual will be selected to enhance their links following the evolutionary process. At time τ, individual v_i has $e_{i,\tau}$ edges, the numbers of edges for the individuals collectively being given by the sequence $e_\tau = (e_{1,\tau}, e_{2,\tau}, \ldots, e_{n,\tau})$. Each vertex v_i is assigned to a target number of links t_i for $i = 1, \ldots, n$ and will attempt to be as close as possible to that target by the end of the interactions, the sequence $t = (t_1, t_2, \ldots, t_n)$ is termed the target sequence. The individuals involved will try to obtain a number of links as close as possible to the target by breaking or forming links between themselves. We assume that $t_i \geq t_j$ for all $i < j$.

The desire for a certain level of social contact, or sociability, is relevant to a wide range of animal species living in groups, including humans, and typically varies between individuals. Studying sociability, it is of interest to try to measure the advantages of social relationships and estimate how long such relationships last, which may be influenced by such factors as an individual's age, gender, dominance etc.

Jackson and Wolinsky [8] discussed the influence of human social networks on behaviour and economic effects. Their research included theoretical analyses of the role of social networks in markets and exchange, they studied a case closely related to ours, where individuals' payoffs depend upon the network, taking into consideration the incentive for individuals to form networks. Related to this is the theory of biological markets created by Noe and colleagues [10, 11]. They established the properties of human markets by relating them to social systems like the peacock's mating system, in which members of one group (often males, as in the case of peacocks) gain by being picked by members of another (females, peahens, for the peacocks).

Among dolphins [12] two degrees of social alliances were noticed when observing dolphins in Shark Bay, Australia, with 14 out of 400 dolphins forming highly unstable alliances, while others formed more solid alliances. According to the research, the complexity of bottlenose dolphins' social relationships may be related to their large brain size. Variation in social alliances might be due to several factors, such as in male baboons (Papio Cynocephalus), where they develop alliances as a conditional strategy, which is mostly utilised by mid-ranking males against high-ranking males. Giraffes [9], tend to develop solid social relations in societies with short path lengths throughout the network.

In sheep [14], a social network was observed and investigated to see the level of their proximity to others when grazing in a group, and whether their tendency to move away from the group was caused by a desire to graze preferred vegetation (long grass). It was discovered that the trade-off between keeping close to one another and grazing further apart on longer grass was relatively small, and so individuals relocating further away mainly reflected other factors, including sociability.

1.2. **Definitions.**

Definition 1.1. The degree of a vertex v_i of a graph G, $deg_G(v_i)$, is the number of links going out of the vertex, which connects the vertex with other vertices.

Definition 1.2. A finite sequence $\bar{e} = \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ of non-negative integers is called a degree sequence of graph G if the vertices of G, $v = v_1, v_2, v_3, \ldots, v_n$, are such that $deg_G(v_i) = e_i$ for all $i = 1, 2, 3, \ldots, n$ (see [15, 13]). We denote by $e = (e_1, e_2, \ldots, e_n)$ a degree sequence of G.

Definition 1.3. A sequence $\bar{e_1}, \bar{e_2}, \bar{e_3}, \ldots, \bar{e_n}$ is graphical if there exists a graph G such that $e(G) = (\bar{e_1}, \bar{e_2}, \bar{e_3}, \ldots, \bar{e_n})$.

Definition 1.4. The distance between the target sequence t and sequence $\bar{e}$ is given by: $Z(t, \bar{e}) = \sum_{i=1}^{n} |t_i - \bar{e}_i|$.
We define the deviation of the graph G as $\mathrm{Z(e(G), t)} = \sum_{i=1}^{n} |t_i - e_i|$.

Definition 1.5. Individual $e_i \in e$ is in deficit when $e_i \neq t_i$.

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Definition 1.6. The score of target sequence t is $s(t) = min_G Z(e(G), t)$; $G \in \boldsymbol{G}$ where $\boldsymbol{G}$ is the set of all graphs. *i.e.* the score is the minimum deviation of all graphical sequences from the target sequence t.

1.3. **Score 1 games.** It is well known that not all degree sequences can be realised as undirected graphs. The Havel-Hakimi algorithm [6, 7] characterises those that can. When the sequence is graphic, all individuals will eventually be on target (i.e. the score will be zero) and this situation is of minimal interest, as then the process will evolve no further. We are more interested in non-graphical sequences where not all the individuals can be on target simultaneously, and more precisely, we want to find the sequences, and associated graphs, with the lowest possible deviation from the original required target, which we classified into a set called the minimal set, denoted by J(min) for the sequences and K(min) for the graphs. To reach the minimal set, the competitors should use rational decisions that prioritise them in getting the desired number of links to achieve their targets. We will consider a particular case of the type of target sequence considered in [5], those as close as possible to the regular graphical sequences, with a score equal to 1. In this model, within the minimal set only one individual is off-target and so only this one individual has an interest in changing its number of links. In particular in any state within K(min) only one individual will attempt to improve its payoff by changing its strategy from any state, and so we know which individual will make the next change irrespective of the order of selected individuals, as all others will **"pass".** We will revisit an example game that was considered in [5].

2. **The sequence 111.** Let us consider the target sequence 111, which has $n = 3$ individuals, represented by the vertices of a graph. It is easy to identify the transitions between the states in this case, and we can see that the transition graph will have 6 vertices which are shown in Figure 1. When the individual is selected and in deficit, it will break one of its links if it has too many links (a Breaker), and it will form a link if it has too few (a Joiner). In particular, the individual to be broken with/ connected to will not be chosen at random, as in the transition probabilities displayed in Section 4.1 in [1], but will be selected strategically.

This particular sequence is of interest as it was (previously considered) the most straightforward **"non-trivial"** sequence for the game-theoretical model. When considering all possible sequences with up to four individuals, we would have two types of situation. Firstly, there are those with straightforward strategies within the minimal set (although see an unusual game from [1] where optimal play can lead to the minimal set being left) in which only one individual ever has a strategic choice to make; for example for the sequence 200 the 0 individuals can have at most a single link, and their optimal strategy is always to break it. Secondly, when this does not occur, we will have a game that has a considerable complexity in analysing the optimal behaviour, for example, the sequence 3210 with J(min)=7 and K(min)=8, which was studied in [4]. It was shown there that there are multiple pure Nash equilibria, and the analysis of mixed equilibria would have been very complex. Similarly, the sequence 2200, which has J(min)=4 but K(min)=16, would lead to the complex computation of the resulting stationary distribution over the 16 states. Sequence 111 is the most straightforward nontrivial sequence found (and likely possible) with J(min)=2 and K(min)=6. Note that the sequence 11111 is

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vastly more complex, as analysis from [5] has shown. In this paper, we show that even 111 is more complicated than previously thought.

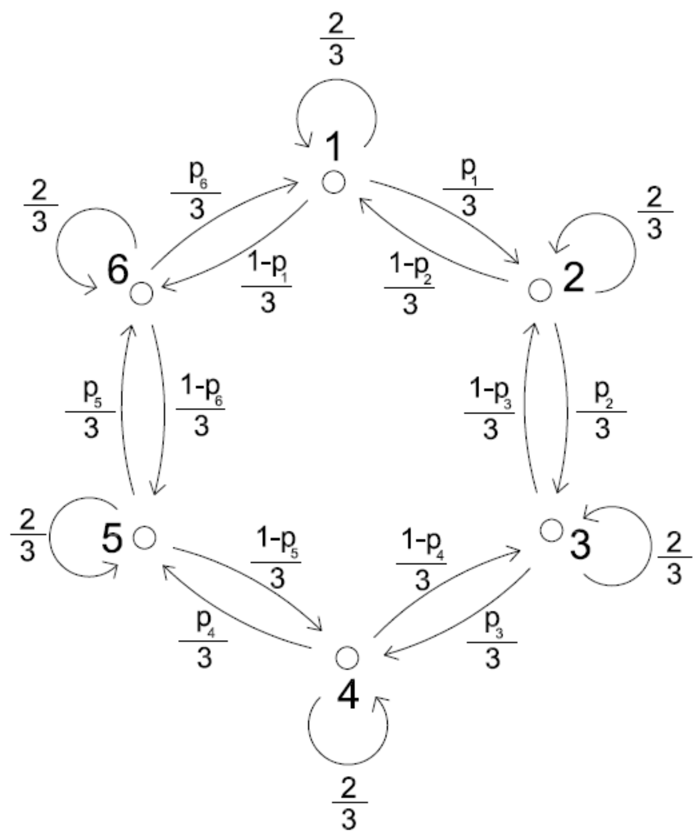

FIGURE 1. Transition graph of the sequence 111 showing the minimal set in every possible move at each of the six vertices. Vertex 1 represents the sequence 211, i.e. individual v_1 is the one in deficit and moves clockwise to sequence 101 with probability $\frac{p_1}{3}$ and anticlockwise to 110 with probability $\frac{1-p_1}{3}$, it will stay in the same state with probability $\frac{2}{3}$. Similarly, vertex 2 represents sequence 101, vertex 3 represents sequence 112, vertex 4 represents sequence 011, vertex 5 represents sequence 121 and vertex 6 represents sequence 110.

2.1. **States and transitions.** Sequence 111 has a transition graph with 6 states. If a vertex not in deficit is selected to update its links, it will make no change. If the vertex in deficit is selected, it will either have two links, one of which it will break, or no links, in which case it will form one of those available. Thus there are two possible transitions.

A collection of probabilities $p_1, \ldots, p_6$ corresponds to a set of strategies to be played by all individuals. So all combinations of probabilities correspond to all possible strategy sets for this game. We define $\pi[i]; i = 1, \ldots, 6$ as the stationary probability distribution for our Markov chain, which will (under the usual assumptions) be the long term probability distribution of the occupancy of the states. The deficit vertex individual in state i will move clockwise with probability p_i or anticlockwise with probability $(1 - p_i)$, the value of p_i being a strategic choice.

The numbered vertices represent the sequence/graph described in the caption of Figure 1 (for example, vertex 1 represents v_1 connected to both other individuals, with the link between the others broken), and the corresponding possible transitions are shown as an arrow moving clockwise and anticlockwise. Transitions occur as follows:
The individual that is potentially to move is chosen at random. This leads to one of the two following cases when in any state i:

1. When the selected individual is neutral, it will decide not to make any transition and to stay at the same position; this occurs with a probability of $\frac{1}{3}$ for each such individual, giving a total probability of 2/3 of the state remaining unchanged, see Figure 1.
2. When the selected individual is in deficit, it will have the choice to move clockwise and (link/break) with the individual who is in deficit at the corresponding vertex with probability p_i or move anticlockwise and (link/break) with the other individual with probability $(1 - p_i)$. We thus have:

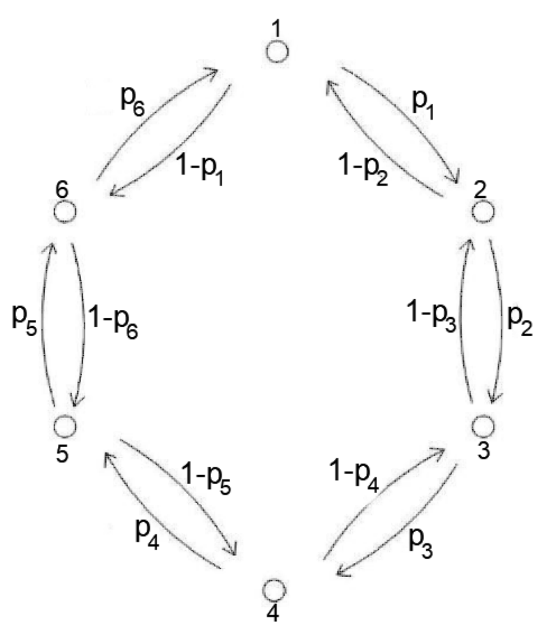

FIGURE 2. Transition graph of the sequence 111 showing the minimal set in every possible move at each of the six vertices. Vertex v_i will move clockwise to vertex v_{i+1} with probability p_i. Vertex v_i will move anticlockwise to vertex v_{i-1} with probability $1 - p_i$.

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$$\pi[1] = \pi[6] * p_6/3 + \pi[1] * 2/3 + \pi[2] * (1 - p_2)/3; \quad (1)$$
$$\pi[2] = \pi[1] * p_1/3 + \pi[2] * 2/3 + \pi[3] * (1 - p_3)/3; \quad (2)$$
$$\pi[3] = \pi[2] * p_2/3 + \pi[3] * 2/3 + \pi[4] * (1 - p_4)/3; \quad (3)$$
$$\pi[4] = \pi[3] * p_3/3 + \pi[4] * 2/3 + \pi[5] * (1 - p_5)/3; \quad (4)$$
$$\pi[5] = \pi[4] * p_4/3 + \pi[5] * 2/3 + \pi[6] * (1 - p_6)/3; \quad (5)$$
$$\pi[6] = \pi[5] * p_5/3 + \pi[6] * 2/3 + \pi[1] * (1 - p_1)/3. \quad (6)$$

This leads to the transition matrix P :

$$P = \begin{bmatrix} \frac{2}{3} & \frac{p_1}{3} & 0 & 0 & 0 & \frac{1-p_1}{3} \\ \frac{1-p_2}{3} & \frac{2}{3} & \frac{p_2}{3} & 0 & 0 & 0 \\ 0 & \frac{1-p_3}{3} & \frac{2}{3} & \frac{p_3}{3} & 0 & 0 \\ 0 & 0 & \frac{1-p_4}{3} & \frac{2}{3} & \frac{p_4}{3} & 0 \\ 0 & 0 & 0 & \frac{1-p_5}{3} & \frac{2}{3} & \frac{p_5}{3} \\ \frac{p_6}{3} & 0 & 0 & 0 & \frac{1-p_6}{3} & \frac{2}{3} \end{bmatrix}. \quad (7)$$

The stationary distribution π can be found by solving $\pi * P = \pi$, in conjunction with the knowledge that

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 = 1. \quad (8)$$

We found the stationary distribution values in terms of $p_1, p_2, p_3, p_4, p_5, p_6$ along with the payoff of the individuals, see the Appendix.

However, only one individual is in deficit, and only that individual would choose to move. Thus the state will be unchanged until eventually that individual is selected. Since staying probabilities are the same in any state, we could effectively assume that the individual is chosen immediately see Figure 2; here, the figure is written using the above equivalence for simplicity.

The payoff for individual v_1 is minus the probability that it is in deficit, i.e. it is minus the probability that the Markov chain is in states 1 or 4. Thus the payoff for v_1 is: $-(\pi[1] + \pi[4])$, and so similarly the payoffs to individuals v_2 and v_3 are $-(\pi[2] + \pi[5])$ and $-(\pi[3] + \pi[6])$ respectively. In [5] three Nash equilibria were found. Since the system will move clockwise if $p_i = 1$ for all i and anticlockwise if $p_i = 0\ \forall i$, in each of these situations each state occurs with a frequency of $1/6$, so the payoff to each individual is $-1/3$. These two sets of strategies are pure Nash equilibria. They also found a mixed Nash equilibrium when $p_i = 1/2\ \forall i$, again with payoffs of $-1/3$ to all individuals. In [5] they showed that when $p_i = 1/2$, the solution is not stable since minor deviations from $1/2$ would subsequently favour strategies that deviate further by running many simulations.

2.2. **Finding Nash equilibria (NE).** We used the payoffs as described above to investigate the Nash equilibria of the game. A Nash equilibrium occurs if and only if no individual has an incentive to change its strategy. For the choice at vertex 1:

$$p_1 \text{is a NE iff } \begin{cases} \frac{d(\pi_1+\pi_4)}{dp_1} \geqslant 0 & p_1 = 0 \\ \frac{d(\pi_1+\pi_4)}{dp_1} = 0 & 0 < p_1 < 1 \\ \frac{d(\pi_1+\pi_4)}{dp_1} \leqslant 0 & p_1 = 1 \end{cases} \quad (9)$$

Recalling that the expressions above in the derivative are minus the payoff to v_1, these conditions ensure that no change in p_1 enables v_1 to gain a higher payoff. Equivalent conditions hold for the other five cases. We found the following Nash equilibria:

1. $p_i = 1$ for all i, every movement (transition) of an individual will be clockwise. For $p_i = 1$ we have

$$\frac{d(\pi_1+\pi_4)}{dp_1} = \frac{d(\pi_1+\pi_4)}{dp_4} = \frac{d(\pi_2+\pi_5)}{dp_5} = \frac{d(\pi_2+\pi_5)}{dp_2} = \frac{d(\pi_3+\pi_6)}{dp_3} = \frac{d(\pi_3+\pi_6)}{dp_6} = \frac{-1}{18},$$

which means that $p_i = 1$ is a Nash equilibrium as in conditions 9.

2. $p_i = 0$ for every i, every movement (transition) of an individual will be anti-clockwise. For $p_i = 0$ we have

$$\frac{d(\pi_1+\pi_4)}{dp_1} = \frac{d(\pi_1+\pi_4)}{dp_4} = \frac{d(\pi_2+\pi_5)}{dp_5} = \frac{d(\pi_2+\pi_5)}{dp_2} = \frac{d(\pi_3+\pi_6)}{dp_3} = \frac{d(\pi_3+\pi_6)}{dp_6} = \frac{1}{18},$$

which means that $p_i = 0$ is a Nash equilibrium from conditions 9.

In cases (1) and (2) each state occurs with frequency 1/6 so the cost to each individual is 1/3. If any individual at any state tried to act differently, then the system will return that individual back to the previous state, and the system will oscillate giving a cost of 1/2 to the individual who switches play. Thus these two sets of choices are strict Nash equilibria.

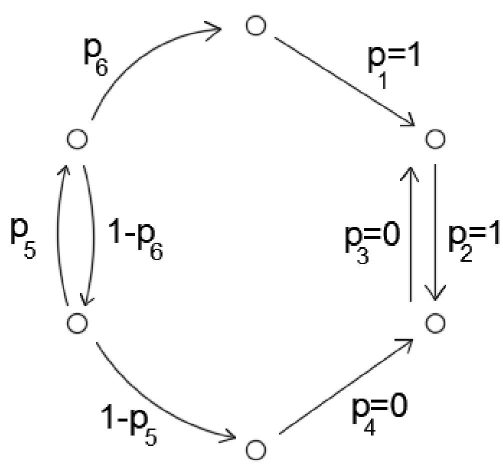

FIGURE 3. Transition graph of the sequence 111 showing the first solution of the cyclic set, solution 4: Straight arrows ↘: when $p_i = 1$, individual v_i will move clockwise. Straight arrows ↖:when $p_i = 0$, individual v_i will move anti-clockwise. Curved arrows: for $0 < p_i < 1$ the movement of individual v_i to the next state could happen in either direction, clockwise with probability p_i or anti-clockwise with probability $1 - p_i$.

3. $p_i = 1/2$ for all i. For $p_i = 1/2$ we have

$$\frac{d(\pi_1+\pi_4)}{dp_1} = \frac{d(\pi_1+\pi_4)}{dp_4} = \frac{d(\pi_2+\pi_5)}{dp_5} = \frac{d(\pi_2+\pi_5)}{dp_2} = \frac{d(\pi_3+\pi_6)}{dp_3} = \frac{d(\pi_3+\pi_6)}{dp_6} = 0,$$

and we again have a Nash equilibrium from conditions 9.

These results reproduce the Nash equilibria from the simpler analyses from [5]. For 4 - 9 we have a set of cyclic solutions which satisfies

$$\frac{d(\pi_1+\pi_4)}{dp_1} = \frac{d(\pi_1+\pi_4)}{dp_4} = \frac{d(\pi_2+\pi_5)}{dp_5} = \frac{d(\pi_2+\pi_5)}{dp_2} = \frac{d(\pi_3+\pi_6)}{dp_3} = \frac{d(\pi_3+\pi_6)}{dp_6} = 0$$

for every solution respectively, and we thus have Nash equilibria as well by conditions 9.

4. $p_1 = 1, p_2 = 1, p_3 = 0, p_4 = 0, 0 < p_5 < 1, 0 < p_6 < 1$, see Figure 3.
5. $0 < p_1 < 1, p_2 = 1, p_3 = 1, p_4 = 0, p_5 = 0, 0 < p_6 < 1$.
6. $0 < p_1 < 1, 0 < p_2 < 1, p_3 = 1, p_4 = 1, p_5 = 0, p_6 = 0$.
7. $p_1 = 0, 0 < p_2 < 1, 0 < p_3 < 1, p_4 = 1, p_5 = 1, p_6 = 0$.
8. $p_1 = 0, p_2 = 0, 0 < p_3 < 1, 0 < p_4 < 1, p_5 = 1, p_6 = 1$.
9. $p_1 = 1, p_2 = 0, p_3 = 0, 0 < p_4 < 1, 0 < p_5 < 1, p_6 = 1$.

Figure 3 illustrates solution 4 showing the movement of each individual. Solutions 5 - 9 are simple rotations of Figure 3.

Let us consider solution 4, where $p_1 = 1, p_2 = 1, p_3 = 0, p_4 = 0, 0 < p_5 < 1, 0 < p_6 < 1$. Here we alternate between states 2 and 3, with a payoff of 0 to v_1 and -1/2 to both v_2 and v_3. We note that this case is a NE due to the following reasoning:
If we are in states 1 and 4 then v_1 would not change strategy, as currently they already receive the maximum payoff of 0 (though actually any change would leave v_1 with payoff 0 too).
If we are in state 5 (state 6) then it makes no difference what strategy the individual in deficit v_2 (v_3) will use as they will end up in states 2 and 3 after enough moves. In state 2, if v_2 picked $p_2 = 0$ instead of $p_2 = 1$ the system will instead alternate between states 1 and 2, leaving the payoff to v_2 unchanged. If some value $0 < p_2 < 1$ was chosen, there would be alternation between state 2 and one of states 1 and 3, with the same result. Similar reasoning holds for the choice of v_3 in state 3. We note from the above that a change in strategy at any vertex does not improve the payoff of the corresponding individual (otherwise it would not be in Nash equilibrium), but it does not make it worse either, and this explains why all the derivatives from condition 9 are equal to 0, as opposed to those in solution 1, for example.

We note that reflecting this figure through its axes, e.g. through the axis from vertex 1 to vertex 4, does not provide any new solutions, as this always leads to one of the other six (for the reflection above, this leads from solution 4 to solution 7).

We have tried to find other sets of Nash equilibria, or to show that they could not exist, but an exhaustive search has proved too complicated. We have been able to demonstrate the absence of Nash equilibria in certain subsets of the parameters, but there are many combinations outstanding.

3. **Discussion.** In this paper, we have considered the dynamically evolving model of [2, 3, 4, 5] where the individuals are competing with each other trying to adjust their links to achieve their desired target. In [5] a particular case of the model was discussed where the sequence has a score equal to 1, this case being the closest to the graphical sequence where the score is equal to zero, and all the individuals are on target. In this paper, we discussed an exceptional example sequence 111 considered in that paper. Broom and Cannings defined three Nash equilibria for this sequence, and at first sight it appeared that these were the only ones, yielding a rare simple but non-trivial case for this game.

For this sequence we considered the individuals' strategies more fully; the strategy of each individual comprises the set of all of their choices at the states for which they are in deficit. Following the chosen strategies, starting at a randomly chosen initial state, transitions representing the individuals' choices will continue indefinitely leading to the distribution over the states following the unique stationary distribution (or if there is not such a distribution, a weighted average over stationary distributions on irreducible subsets of the states). Thus we can find the payoffs for any set of strategies, and thus potentially Nash equilibria for the game.

We identified a stable sets of strategies which led to another six cyclic Nash equilibria for the game in addition to the original three solutions, but found no other solutions. This does not mean that no such solutions exist, and a complete analysis of the game would be very complicated. Thus although the game over this sequence initially seemed relatively simple, in fact there has turned out to be significant complexity. All target sequences thus seem to yield on the one hand trivial games over the minimal set, where the number of players which can be in deficit is 0 (i.e. graphical sequences with no game at all), 2 (leading to simple pairwise swaps between the individuals) or more than two but involving a collection of such simple swaps (such as the sequence 200); on the other hand we obtain complex games involving three or more interacting individuals, with our example as potentially the simplest of these.

An interesting question is whether this complexity is inherent in such games, or relies on specific assumptions of our model. In future work we will consider variants of this game which might be more amenable to analysis. In particular we are considering an alternative model, where both individuals must agree to form a link before it can be established, but where either individual can break a link.

Appendix. In the calculations in this section we use the following alternative representation to minimise the number of subscripts used: $x = p_1, y = p_2, z = p_3, f = p_4, g = p_5, h = p_6$. Substituting these values in the row vector $\pi * P = \pi$ where π is the stationary distribution, and replacing the last (redundant) row by the condition that the row vector entries (probabilities) are non-negative and sum to 1, yields

$$\begin{bmatrix} -1 & -y+1 & 0 & 0 & 0 & h \\ x & -1 & -z+1 & 0 & 0 & 0 \\ 0 & y & -1 & -f+1 & 0 & 0 \\ 0 & 0 & z & -1 & -g+1 & 0 \\ 0 & 0 & 0 & f & -1 & -h+1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solving this set of equations by the inverse matrix method we obtained the values of the stationary distribution $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ and π_6 in term of x, y, z, f, g and h as follows. Denoting the common denominator term for all components of the stationary distribution by

$$A = (4f - 4gf + 4g - 2hf + 2ghf - 4gh + 4h - 2fx + 2fgx - 2gx + 2hfx - 2ghxf$$
$$+ 2ghx - 4hx + 4x - 2yf + 2gyf - 2gy + 2ghy - 2hy + 2fxy - 2fgxy + 2gxy - 2ghxy$$
$$+ 2hxy - 4xy + 4y - 4fz + 2fgz - 2gz + 2fhz - 2fghz + 2ghz - 2hz + 2fxz - 2fhxz$$
$$+ 2hxz - 2xz + 2fyz - 2fgyz + 2gyz - 2ghyz + 2hyz - 2fxyz - 2hxyz + 2xyz - 4yz + 4z - 6),$$

the stationary distribution terms are:

$$\pi_1 = \frac{f-fg+g-gh-fy+fgy-gy+ghy+y-fz+fgz-gz-fghz+ghz+fyz-fgyz+gyz-ghyz-yz+z-1}{A}$$

$$\pi_2 = \frac{f-fg+g-fh+fgh-gh+h+fhx-fghx-hx-fz+fgz-gz+fhz-fghz+ghz-hz-fhxz+hxz+z-1}{A}$$

$$\pi_3 = \frac{f-fg+g-fh+fgh-gh+h-fx+fgx-gx+fhx-fghx+ghx-hx+x+fxy-fgxy+gxy-ghxy-xy-1}{A}$$

$$\pi_4 = \frac{g-gh+h-gx+ghx-hx+x-gy+ghy-hy+gxy-ghxy+hxy-xy+y+gyz-ghyz+hyz-hxyz-yz-1}{A}$$

$$\pi_5 = \frac{h-hx+x-hy+hxy-xy+y-fz+fhz-hz+fxz-fhxz+hxz-xz+hyz-fxyz-hxyz+xyz-yz+z-1}{A}$$

$$\pi_6 = \frac{f-fg-fx+fgx+x-fy+fgy+fxy-fgxy-xy+y-fz+fxz-xz+fyz-fgyz-fxyz+xyz-yz+z-1}{A}.$$

For the general game the payoff of the individuals is defined in [4] as the negative of their expected long term deviation, thus the payoff to individual v_i is:
$R_i(U_X) = -\sum_X \epsilon_i(X) * \pi(X)$
where $\epsilon_i(X)$ is the deviation of v_i in state X, and $\pi(X)$ is the stationary distribution over X.
We then have the following payoffs of the individuals in the sequence 111: the payoff of v_1 is $-(\pi_1+\pi_4)$, the payoff of v_2 is $-(\pi_2+\pi_5)$ and the payoff of v_3 is $-(\pi_3+\pi_6)$.

We then found the derivatives of the negative of the payoffs of each of the three individuals, each the sum of two π_i terms, at the two states where they are in deficit with respect to the variable that govern the transition at the corresponding state. Starting by differentiating $(\pi_1 + \pi_4)$ with respect to x we get:

$$\frac{d(\pi_1+\pi_4)(x,y,z,f,g,h)}{dx} = \frac{\alpha}{\beta},$$

where
$\alpha = -((-1+h+z-hz+hy(-1+g+z)+f(-1+h+y)(-1+g+z)-y(-2+g+2z))((1+g(-1+h))(-1+y)(-1+z)+f(-1+g+y-gy+z-yz+g(-1+h+y)z)))$
and
$\beta = 2\{3-2h-2x+2hx+f(-1+g)(2+h(-1+x)+x(-1+y)-y)-2y+hy+2xy-hxy+g(-1+h)(2+x(-1+y)+y(-1+z)-z)+(2+h(-1+x)-x)(-1+y)z+f(2+h(-1+x)-x+(-1+x)y+g(-1+h+y))z\}^2$

In the same manner we found $\frac{d(\pi_1+\pi_4)(x,y,z,f,g,h)}{df}$. We notice that finding the derivatives is a complicated procedure which produces a long formula. Therefore we used the symmetry of this dynamical system which made the calculations easier. Due to the symmetry in the dynamical system we will have:

$$\frac{d(\pi_1+\pi_4)(x,y,z,f,g,h)}{dx} = \frac{d(\pi_2+\pi_5)(h,x,y,z,f,g)}{dy} = \frac{d(\pi_3+\pi_6)(y,z,f,g,h,x)}{dh}$$
$$= \frac{d(\pi_1+\pi_4)(x,y,z,f,g,h)}{df} = \frac{d(\pi_2+\pi_5)(h,x,y,z,f,g)}{dg} = \frac{d(\pi_3+\pi_6)(y,z,f,g,h,x)}{dz}.$$

We thus have all of the terms required to find Nash equilibria following Equation 9.

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