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Modelling Dominance Hierarchy formation as a Multi-player game

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Animals who live in groups need to divide available resources amongst themselves. This is often achieved by means of a dominance hierarchy, where dominant individuals obtain a larger share of the resources than subordinate individuals. This paper introduces a model of dominance hierarchy formation using a multi-player extension of the classical Hawk–Dove game. Animals play non-independent pairwise games in a *Swiss tournament* which pairs opponents against those which have performed equally well in the conflict so far, for a fixed number of rounds. Resources are divided according to the number of contests won. The model, and its emergent properties, are discussed in the context of experimental observations.

1. Introduction

1.1. DOMINANCE HIERARCHIES

Many animals spend important parts of their lives or their entire lives in groups, where the individuals do not have separate territories, but occupy a combined territory together. A common feature of such groups is the presence of dominance hierarchies, where the animals sort themselves into a preference order for feeding, mating etc., see Alcock (1993). Two important questions relating to dominance hierarchies are

1. How are these hierarchies formed?
2. Once formed, are they stable and if so how are they maintained?

There is a reliable body of theory which goes a long way to answering the second question. Linear hierarchies (defined below) tend to be stable. Experiments have been carried out where individual fowl have been removed from the population, and reintroduced days later, where they re-adopt their former position, often without any conflict (Klopfer, 1973). On the other hand, Guhl (1968) has reported that the correlation in position is small when the whole group establishes a hierarchy at two distinct time points. These experiment suggests that it is not intrinsic properties of the individuals which determine the hierarchy, but rather the context and overall interactions within the group. There is evidence that dominance hierarchies are maintained by recognizing dominant and non-dominant individuals e.g. by scent in fish (Todd *et al.*, 1967) or by voice in birds (Lemon, 1967). In monkeys coalitions between non-related individuals have been observed, see Chapais *et al.* (1991). An important group of theoretical papers, originating with Vehrencamp (1983),

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have considered this problem in relation to the concept of *reproductive skew*, which is concerned with how the resources within a group of animals are divided. This division is often very uneven, high reproductive skew (e.g. Rood, 1980), so that the top few animals receive almost all of the pay-off. A model was developed in Keller & Reeve (1994) where subordinate individuals are given limited mating rights (stay and peace incentives) in order not to challenge for dominance or leave the group.

Our interest here is in the first question and we develop a model of how such a hierarchy might be formed, and discuss its implications and efficacy. Our approach is to assume that the individuals are initially undifferentiated in any way that would affect their competitive ability. In contrast to the model of Bonabeau *et al.* (1996) we assume that an individual's probability of winning is unaffected by its history, at least during the main part of the hierarchy formation. They choose strategies and win or lose, within a framework which eventually establishes a partial ordering of the individuals.

1.1.1. *Forming Hierarchies*

In long-term hierarchies where animals spend their whole lives in the same group, for example colonies of monkeys, offspring often attain the social rank of their mother (Marsden, 1968). Similar behaviour has also been recorded in hyenas (Frank, 1986). Thus as individuals join the group one by one, they slot into an already established hierarchy at a pre-determined place.

On the other hand, many species form and reform groups throughout their lives and dominance hierarchies will be established from scratch. This happens for example, when groups form by a large number of adults coming together almost simultaneously, for example birds gathering in a lek (Hoglund & Alatalo, 1995) at the beginning of the breeding season.

There is considerable experimental work investigating this establishment of hierarchies. Interactions have been set up in the laboratory, and the "winner" and "loser" observed in a sequence of encounters. There are two important findings from this work:

1. Hierarchies tend to be approximately linear (well-ordered). A hierarchy is linear if one can number the individuals 1, 2, ..., n in such a way that if $i<j$ then i is dominant to j.
2. There may be correlations between the outcomes of successive contests. A loser may have an increased probability of losing its next contest also, and this is not explicable by the fact that a loser is likely to be weaker and therefore less likely to win in general.

Linearity. Chase & Rohwer (1987), for example, brought together groups of 12–17 Harris Sparrows *Zonotrichia quereula*, and studied the interactions over time. They were interested primarily in the linearity of the hierarchy formed, and more particularly in the triads. A triad is created when three individuals A, B and C have interacted pairwise; thus A has contested with B, B with C and A with C. Denoting the event that A beat B by $A\rightarrow B$ such a triad is then termed transitive if, for example, $A\rightarrow B$, $B\rightarrow C$ and $A\rightarrow C$, whereas if $A\rightarrow B$, $B\rightarrow C$, and $C\rightarrow A$ the triad is cyclic. Of the 611 triads observed by Chase and Rohmer, 540 were transitive and 41 cyclic, vis-a-vis expectations of 458 and 153 if outcomes were random. They further observed that the hierarchies formed were significantly more linear than would occur by chance.

The method we use below also models the formation of a hierarchy through pairwise interactions, but in a highly structured way. Full details are given below but we essentially suppose that individuals come together to establish the hierarchy and do so by a series of contests, at each stage competing with another individual who has had a similar success rate. We envisage this happening due to some additional feature of the environment, perhaps spatial. For example, suppose that there is a habitat in which higher positions (in altitude) are favoured, as in the carpenter bee *Xylocopa* (*Neoxylocopa*) *varipuncta*, (Alcock & Smith, 1987). Initially, bees might aggregate at some intermediate height, winners moving up and losers down so that at a particular level at any point in time are individuals who have had comparable success in the past. The outcome will be the division of individuals into sets with equal status. We do not have every pairwise dominance resolved, and so the issue of linearity does not arise quite in the manner defined above. Here if $A > B$, meaning A has higher status than B, and $B > C$ then necessarily $A > C$. One should perhaps note that once the basic competition is over, individuals may still be expected to compete within the level that they have achieved and thus refine the detail of the hierarchy. Thus, one would expect to see at that stage primarily interactions between individuals of similar rank, and few if any, between individuals very disparate in rank, which accords with observation.

On the other hand, it is possible that during the basic competition intransitive triads may occur. In the simple case where we have only four individuals there are essentially only two patterns of outcomes,

1. $A \rightarrow B$ and $C \rightarrow D$ in round 1, $A \rightarrow C$ and $B \rightarrow D$ in round 2.
2. $A \rightarrow B$ and $C \rightarrow D$ in round 1, $A \rightarrow C$ and $D \rightarrow B$ in round 2

and neither of these has a triad. With more individuals triads may arise. With eight individuals there are 288 permutationally distinct configurations. Within these there can be no cyclic triads, though exceptionally a four cycle can occur. In general, the number of triads will be small, and the chance of these being cyclic also smaller than in the all-play-all situation (where the probability is 0.25). Essentially this is because in many triads the highest ranked individual will have finished high up the hierarchy and so there is a greater than 0.5 chance that he won both contests in the triad. For example, the chance of a triad involving the top player being cyclic is zero, and this is also true for the individual who lost to the top player in the last round. The bottom player and his last round opponent are also excluded from being in a cyclic triad. In general, the outcomes will be fairly linear in the conventional sense.

We note a detail of interest in the case of four players; the same phenomenon applies more generally. Both possible outcomes divide the individuals into three levels, won twice, won once and lost once, and lost twice. However, there is a difference of some possible significance. In the first outcome, the two players, B and C, at the intermediate level have not played each other, whereas in the second outcome, C and D are at the intermediate level but $C \rightarrow D$. One could argue that this would therefore resolve the hierarchy into four levels rather than only three. This phenomenon will be less marked with larger numbers of individuals.

Correlations. A second major issue in considering the formation of dominance hierarchies is the possible existence of correlations between the outcomes of the successive contests of an individual. These effects have been investigated in many studies (e.g. Frey & Miller, 1972; Chase *et al.*, 1994; Hsu & Wolf, 2001). The results tend to suggest that losing increases the chance of losing the next contest, whereas the effect of winning is less clear cut. We have not included this feature in our model. As stated above individuals contest with others who have a similar overall success rate and the outcome is not affected by the specific sequence of wins and losses they have experienced. On average in our model a winner wins his next contest with probability one-half, and so does a loser. There are at least two distinct ways in which one could justify the absense of this effect in a population,

1. The contests take place sufficiently separated in time that there is no carry through from the previous contest or, equivalently, that the individuals have short memories or rapid recovery capacity.
2. The individuals have very good memories so that they can take into account their whole history, or that their perception of their abilities is very strongly linked to their current position (e.g. altitude on the hill for carpenter bees) so that their route to that position is not relevent (a Markovian assumption).

1.2. GAME THEORY

Game theory originated in the area of economics with von Neumann & Morgenstern (1944). Other important texts include Binmore (1992) and Harsanyi & Selten (1988). Game theoretic methods have been applied to many biological situations, especially those where animals are in conflict with each other for some resource, for example food or mates, or in our case status, which in turn gives precedence over others for a range of resources. Examples include the modelling of signalling behaviour (Grafen, 1990a, b) and the modelling of anti-predator vigilance (McNamara & Houston, 1992; Pulliam *et al.*, 1982).

An important early application of game theory to biology was by Maynard Smith & Price (1993), which introduced the Hawk–Dove game which forms the starting point of this paper. This idea was developed further in Maynard Smith (1982). It was shown in Maynard Smith & Price (1973) how game theory could explain the apparently contradictory behaviour of heavily armed animals which did not use their armaments effectively, but only engaged in ritualistic contests. The basic structure of the Hawk–Dove game is as follows;

Animals compete for a reward of value $V > 0$. They are all of the same strength/size and each animal is equally likely to meet all possible opponents. There are two available *pure strategies*, aggressive "Hawk" and non-aggressive "Dove". The reward or *pay-off* to players is described below.

If a Hawk meets a Dove, the Dove flees (receiving nothing) and the Hawk picks up the reward V. If two Doves meet they engage in a ritualistic display, each winning the reward with probability 1/2, and thus each receives expected pay-off $V/2$.

If two Hawks meet they fight, each winning the reward with probability 1/2, the loser receiving an injury of cost $C > 0$ (reward $-C$) and thus receive expected pay-off $(V - C)/2$.

The pay-off information is generally summarized in the *pay-off matrix* $\mathbf{A} = (a_{ij})$, where a_{ij} is the pay-off to an individual playing strategy i against one playing j. Thus for the Hawk–Dove game, where Hawk is strategy 1 and Dove is strategy 2, the pay-off matrix is

$$\begin{bmatrix} \frac{1}{2}(V - C) & V \\ 0 & \frac{1}{2}V \end{bmatrix}. \tag{1}$$

Suppose that the proportion of Hawk players in the population is p. Maynard Smith (1982) showed that the population is in stable equilibrium [i.e. p is an *Evolutionarily Stable Strategy* (*ESS*)] if and only if

$$p = \frac{V}{C}, \;\; V < C, \;\; p = 1, \;\; V \geqslant C.$$

In particular, if C is much larger than V, e.g. cost of death vs. a mating opportunity, the strategy Hawk would be very rare.

The Hawk–Dove game, and most other game theory models, involve only two players. There may be many animals in the population, but the model assumes that the result of a contest between one pair of animals has no bearing upon that of another pair. When modelling the formation of a dominance hierarchy this assumption is not reasonable. The animals sort themselves into a ranking order so that every game which affects an animal's position in the order affects all of the animals in some way; it is certainly likely to influence which opponent an animal is likely to face next (winners will tend to fight other winners, losers other losers, any potential mismatched contests are likely to end with the weaker animal backing down). Thus, the game will be multi-player in character.

Together with G.T.Vickers, we have recently published a series of papers developing multi-player models. These have been both of the general variety where many individuals compete simultaneously (Broom *et al.*, 1997b) and with a structure in which pairwise games are played, to allow for non-independence (Broom *et al.*, 1996, 1997a, 2000a, b). We take the second approach here, assuming that individuals compete in pairwise contests of Hawk–Dove type, but that the results of these contests determine which opponents are likely to be faced in the future. Thus we create a tournament structure.

1.2.1. *Tournament Models*

A tournament can be organized in many ways but usually there are a number of rounds, each round consisting of the pairing of some or all of the contestants. The two most familiar structures are the knockout tournament and the round-robin (or all-play-all) tournament. In the knockout tournament, the 2^n players are paired in the first round (either randomly or as the outcome of some preliminary contest), the 2^{n-1} losers being eliminated, while the 2^{n-1} progress to the next round. This process continues until a single undefeated individual results (the winner). Thus there are a total of $2^n - 1$ contests, since each contest produces a loser and each contestant, other than the winner, loses once. In the round-robin tournament every possible pairing occurs so there is an intrinsic difficulty in defining an individual's ultimate rank, unless there is a linear outcome. The number of contests in this case is $2^{n-1} * (2^n - 1)$ being just the number of pairings possible. The Swiss tournament which we define below has a number of contests equal to $2^n * n$ which is intermediate between the knockout and round-robin.

Knockout tournaments have been considered in Broom *et al.* (2000a, b). In specifying the details of a tournament one has to define the repetoire of the individuals. In Broom *et al.* (2000a) each individual was constrained to play either Dove or Hawk in every round it reached, whereas in Broom *et al.* (2000b) individuals could choose round by round. It is this latter approach which we adopt here. A knockout tournament is effective in producing a single winner in relatively few contests, and so will be appropriate where there is essentially only a reward for the α-animal, and indeed emerges from within our model below. However, when rewards are obtained for intermediate ranks the knockout tournament may be inappropriate.

Mesterton-Gibbons & Dugatkin (1995) model hierarchy formation using a round-robin model of independent pairwise conflicts for small groups $n \leqslant 10$. It seems likely that such all-play-all contests are only practical for very small groups (otherwise the number of fights would be large and lead to energy waste and injury). We consider the *Swiss tournament* common in Chess and described in detail in Section 1.3, which establishes an order with far less conflicts. It achieves a rough ordering very quickly, which then might be further resolved, and so might be efficient when there is a spread of rewards. We compare very briefly the performance of a variety of different "tournaments" in the Discussion.

1.3. SWISS TOURNAMENTS

Swiss tournaments are commonly used in chess competitions. All players play in every round (provided that the number of competitors is even), players being matched with opponents on the same score, if possible. If this cannot be done, opponents are paired as closely as possible (there are precise rules for how this should be done). There are three main advantages of organizing tournaments in this way, the second and third of which are pertinent to the modelling of dominance hierarchies;

(1) In knockout tournaments, losers are eliminated after the first game, which is unsatisfactory for players who will have paid a (possibly substantial) fee to enter the tournament. Also, many tournaments are played over a single weekend when only five or six games of chess can be played, so that a round-robin tournament could only involve a small number of competitors (typical Swiss tournaments involve about 40).

(2) After a couple of rounds, players tend to play opponents at roughly their own playing level, since such players will generally be on or near the same score. Thus games are roughly even. In the animal world, individuals generally fight opponents of roughly equal strength (though we assume here that individuals only differ in their strategies and previous success and not in their strength), since if it is obvious who will win before the contest, the inferior animal will usually back down.

(3) Swiss tournaments establish a reasonably accurate order according to ability in a very short number of contests. If there were no draws, a winner could be established in n rounds if the number of players is 2^n or less, although the ordering in the middle of the field is less precise than at either end. This feature is important in the real world to minimize the risk of injury and the amount of energy wasted.

We do not claim that animals will actually organize such structured tournaments, but the tendency to play compatible opponents and the necessity to minimize the number of fights means that such a tournament might be a reasonable approximation.

In this paper, we explore the strategic behaviour in an n-round Swiss tournament. Section 2 introduces a minor generalization of the Hawk–Dove game suitable for embedding in a multi-round contest. After specifying the ESS for such a contest we derive two results relating pay-offs to plays. Section 3 introduces the Swiss tournament formally, each round consisting of contests played as per Section 2. We derive a number of results which simplify the task of finding potential ESSs. Section 4 considers some examples, i.e. choices of the rewards and costs. Specifically, we consider the cases where the reward doubles with each increment of unity in the number of wins, and the case where only the "overall winner" (the individual which has won every contest) receives a reward. Section 5 addresses the question of what strategies can exist. Earlier the set of strategies was shown to be at most $(n+1)!$ in number. It is proved that these are all achievable. We discuss in particular the cases $n=2$ and 3 and give diagrams of parts of the strategy space, which show that this is rather complex. Section 6 discusses certain issues relating to comparisons of the Swiss, knockout, all-play-all and other tournaments.

2. Preliminaries

In the classical Hawk–Dove game, animals compete for a reward value V, the loser receiving 0 (if the contest was Hawk–Hawk the loser also incurs a cost C). We generalize this game so that winners receive W and losers receive L. This is important for the sections which follow. In such a contest the pay-off matrix becomes

$$\begin{bmatrix} \frac{1}{2}(W+L-C) & W \\ L & \frac{1}{2}(W+L) \end{bmatrix}. \tag{2}$$

There is always a unique ESS of this game. If p is the probability of playing Hawk in the ESS, then

(i) $p=0$ if $W-L<0$. The pay-off to this strategy which is just "play Dove" is

$$\tfrac{1}{2}(W+L).$$

(ii) $p=1$ if $W-L\geqslant C$. This strategy is "play Hawk" and the pay-off is

$$\tfrac{1}{2}(W+L-C).$$

(iii) If $0\leqslant W-L<C$, then

$$p=\frac{W-L}{C}$$

and the pay-off is

$$\frac{1}{2}(W+L)-\frac{1}{2C}(W-L)^2.$$

In general, the pay-off is

$$\tfrac{1}{2}(W+L-p^2C),$$

and

$$p=mid\left(0,\frac{W-L}{C},1\right),$$

where $mid(a,b,c)$ is the second largest value of a,b and c. It turns out that in our model, as in the original Hawk–Dove game, case (i) never occurs [although $p=0$ is possible from the lower limit of case (iii)], so that the pay-off can be rewritten as

$$\frac{1}{2}\left(W+L-\frac{1}{C}\{min(C,(W-L))\}^2\right).$$

To facilitate later proofs, we define the function $R(x,y)$ as

$$R(x,y)=\frac{1}{2}\left(x+y-\frac{1}{C}\{min(C,(x-y))\}^2\right)$$

for $x\geqslant y$.

Thus $R(x,y)$ is the expected pay-off at an ESS, when $W=x$ and $L=y$. We also prove two results which are used later in the paper.

Result 2.1. *If* $x\geqslant y$ *then* $x\geqslant R(x,y)\geqslant y$.

Thus the expected pay-off in any contest lies between the pay-off to the winner and the pay-off (*excluding the cost*) to the loser.

Proof.

$$\begin{aligned} R(x,y) &= \frac{x+y}{2} - \frac{1}{2C}\{min(C,(x-y))\}^2 \\ &\geqslant \frac{x+y}{2} - \frac{1}{2C}\{C(x-y)\} = y, \end{aligned}$$

$$\begin{aligned} R(x,y) &= \frac{x+y}{2} - \frac{1}{2C}\{min(C,(x-y))\}^2 \\ &\leqslant \frac{x+y}{2} \leqslant x. \quad \square \end{aligned}$$

Result 2.2. *If* $x \geqslant y \geqslant z$ *and* $y - z \geqslant C$ *then* $R(x,y) - R(y,z) \geqslant C$.

Here an individual is looking two contests ahead. The rewards for winning twice will be x, for winning once and losing once y, and for losing twice z. The reward for winning exactly one contest, y, is at least C greater than that for winning no contests, z. If the individual wins the first contest then his expected payoff under the ESS in the second contest is $R(x, y)$, while if he loses it is $R(y, z)$. We prove the difference between these quantities is at least C.

Proof.

$$\begin{aligned} R(x,y) - R(y,z) &\geqslant y - \tfrac{1}{2}(y + z - C) \\ &= \tfrac{1}{2}(y - z + C) \geqslant C. \quad \square \end{aligned}$$

3. The Swiss Tournament Model

We consider a game where there are 2^n players, and every pairwise game ends in a win for one of the players (no draws). n rounds are played, thus giving a unique winner, and ensuring every player is always paired with an opponent on the same score.

Initially there are 2^n players of equal strength on 0/0 (i.e. 0 wins from 0 contests). These are paired, and the games are played. There are now 2^{n-1} players on 0/1 and 2^{n-1} on 1/1. They are repaired with players on the same score giving 2^{n-2} on 0/2, 2^{n-1} on 1/2 and 2^{n-2} on 2/2. In general after j rounds there are

$$\binom{j}{i}2^{n-j}$$

players on i/j $i = 0, \ldots, j$.

Suppose that the reward for finishing on a score of i/n is V_i $i = 0, \ldots, n$ such that $V_0 \leqslant V_1 \leqslant \cdots \leqslant V_n$.

Players play a Swiss tournament where the pairwise games are Hawk–Dove, the winner gaining an extra point and the loser staying on the same score.

We find the optimal play at every stage of the game, i.e. on every score before the competition ends i/j $0 \leqslant i \leqslant j \leqslant n-1$, by finding the optimal play and the expected pay-off for a player on the score i/j, conditional upon the optimal play and expected pay-offs to a player on the scores $i/(j+1)$ and $(i+1)/(j+1)$, which are the two possible positions the player could be in the next round.

Consider the *overall strategy* that a player employs, i.e. a choice of strategy for every possible position i/j. For the overall strategy to be optimal, it is clear that the final round play must be as in the ESS of the original Hawk–Dove game (otherwise it is invaded by a strategy playing as it does except playing the Hawk–Dove ESS in the final round). Thus, the strategy must assume that this is how play occurs in the final round (which is true if every player plays it) and on this assumption the expected reward for each possible score immediately before the final round can be found. This in turn gives us the optimal play for the previous round, etc.

It is easy to show that this overall strategy cannot be invaded by any strategy whose play differs only in a single round, but it may be invaded by a strategy which differs in more than one round. Note, however, that there is always some evolutionary pressure towards this optimal strategy, since any other strategy can be invaded by a strategy identical except in the last round it does not play as the optimal strategy, at which point the invading strategy plays as the optimal strategy. Thus, it can be invaded by a strategy one step closer to the optimal. This also means that the optimal strategy is the only one which cannot be invaded by a strategy a single change away. On the assumption that mutations produce strategies which closely resemble the original, the optimal strategy is the only one which is resistant to invasion.

Define the following:

(1) W_{ij} is the expected reward to a player with a score of i/j (i.e. at the end of round j). Thus $W_{in} = V_i \forall i$.

(2) $\mathbf{M_{ij}}$ is the pay-off matrix for a player with a score of i/j playing in round $j+1$. The player has expected reward $W_{i(j+1)}$ if it loses and $W_{(i+1)(j+1)}$ if it wins. So we have

$$\mathbf{M_{ij}} = \begin{bmatrix} \frac{1}{2}(W_{(i+1)(j+1)} + W_{i(j+1)} - C) & W_{(i+1)(j+1)} \\ W_{i(j+1)} & \frac{1}{2}(W_{(i+1)(j+1)} + W_{i(j+1)}) \end{bmatrix}. \qquad (3)$$

Result 3.1. *$i \leqslant k \leqslant j \Rightarrow W_{ij} \leqslant W_{kj}$.*

Thus the expected reward for a player with a score k is at least as good as for a player with a lower score i, at the end of round j.

Proof. We know that $V_0 \leqslant V_1 \leqslant \cdots \leqslant V_n$. Since $W_{i(n-1)} = R(V_{i+1}, V_i)\ \forall i$, Result 2.1 implies that $W_{0(n-1)} \leqslant W_{1(n-1)} \leqslant \cdots \leqslant W_{(n-1)(n-1)}$. An inductive argument shows that $W_{0j} \leqslant W_{1j} \leqslant \cdots \leqslant W_{jj}$ for all j, thus proving the result. □

(3) Further define p_{ij} as the probability of playing Hawk in the ESS of $\mathbf{M_{ij}}$, giving

$$p_{ij} = mid\left(0, \frac{W_{(i+1)(j+1)} - W_{i(j+1)}}{C}, 1\right)$$

and

$$W_{ij} = \tfrac{1}{2}\left(W_{(i+1)(j+1)} + W_{i(j+1)} - p_{ij}^2 C\right) \ldots (1).$$

Result 3.1 implies that

$$p_{ij} = min\left(\frac{W_{(i+1)(j+1)} - W_{i(j+1)}}{C}, 1\right) \ldots (2)$$

and

$$W_{ij} = R(W_{(i+1)(j+1)}, W_{i(j+1)})$$

for all $i \leqslant j \leqslant n-1$.

Result 3.2a. *$W_{ij} \leqslant W_{(i+1)(j+1)}$ if $i \leqslant j \leqslant n-1$.*

If you win a game your expected pay-off cannot decrease.

Proof. $W_{ij} = R(W_{(i+1)(j+1)}, W_{i(j+1)}) \leqslant W_{(i+1)(j+1)}$ since $R(x,y) \leqslant x$ for all $x \geqslant y$ (from Result 2.1). □

Result 3.2b. *$W_{ij} \geqslant W_{i(j+1)}$ if $i \leqslant j \leqslant n-1$.*

If you lose a game your expected pay-off cannot increase (even ignoring any cost which might have been incurred).

Proof. $W_{ij} = R(W_{(i+1)(j+1)}, W_{i(j+1)}) \geqslant W_{i(j+1)}$ since $R(x,y) \geqslant y$ for all $x \geqslant y$ (again from Result 2.1). □

The above results are intuitively what we would expect; a win improves our position, a loss makes our situation worse and a high score is better than a low one. The following result is less intuitive;

Result 3.3. If $p_{i(j+1)} = 1$ *then* $p_{ij} = 1$.

Equivalently if $p_{ij} < 1$ then $p_{i(j+1)} < 1$, so that if your optimal play is a mixed strategy, and you lose the contest, the optimal play for the next round is also a mixed strategy. Note that the p_{ij} are not necessarily monotonically decreasing with j for a given i (see Example 1).

Proof. $p_{i(j+1)} = 1 \Rightarrow W_{(i+1)(j+2)} - W_{i(j+2)} \geqslant C$. We know from Result 3.1 that $W_{(i+2)(j+2)} \geqslant W_{(i+1)(j+2)} \geqslant W_{i(j+2)}$. Using Result 2.2 this implies that

$$R(W_{(i+2)(j+2)}, W_{(i+1)(j+2)}) - R(W_{(i+1)(j+2)},$$

$$W_{i(j+2)}) \geqslant C \Rightarrow W_{(i+1)(j+1)} - W_{i(j+1)} \geqslant C$$

$$\Rightarrow p_{ij} = 1. \quad \square$$

Thus if an animal starts off playing aggressively, if it incur's a few losses the strategy should become more cautious. If it starts playing cautiously it should not then become aggressive if it loses.

4. Examples

4.1. EXAMPLE 1. DOUBLING REWARDS

Consider the game with $C = 5, n = 8, V_i = 2^i \; i = 0, \ldots, 8$.

This is an example of high reproductive skew, as the rewards for the top animals are far larger than for the lower animals. Successive values of W_{ij} and p_{ij} are found using the recurrence relations from eqns (1) and (2), namely

$$p_{ij} = min\left(\frac{W_{(i+1)(j+1)} - W_{i(j+1)}}{5}, 1\right),$$

$$W_{ij} = \tfrac{1}{2}\left(W_{(i+1)(j+1)} + W_{i(j+1)} - 5p_{ij}^2\right).$$

For example, the expected rewards for the scores 1/7, 2/7 and 3/7 are $W_{17} = 2.6, W_{27} = 4.4$ and $W_{37} = 9.5$.

$$W_{37} - W_{27} \geqslant 5 \Rightarrow p_{26} = 1 \Rightarrow W_{26}$$

$$= \tfrac{1}{2}(9.5 + 4.4 - 5) = 4.45.$$

$$W_{27} - W_{17} < 5 \Rightarrow p_{16} = \frac{W_{27} - W_{17}}{5}$$

$$= \frac{4.4 - 2.6}{5} = 0.36 \Rightarrow W_{16}$$

$$= \frac{1}{2}(4.4 + 2.6 - 5(0.36)^2) = 3.176.$$

The values of W_{ij} are given in Table 1. The values of p_{ij} are given in Table 2.

In Example 1, the rewards are high only for those on the highest scores, most players receiving relatively low scores. This is reasonable, since in many species the number of males with mating rights, for example, tends to be small. The optimal play is to play Hawk when you have a chance of occupying one of these top spots (early in the contest, or later if you have a high score), and to play a more cautious mixed strategy otherwise. Note the non-monotone behaviour of p_{ij} in rows $i = 0$ and 1.

TABLE 1
Expected pay-offs to players on score i/j

i/j	0	1	2	3	4	5	6	7	8
8	—	—	—	—	—	—	—	—	256
7	—	—	—	—	—	—	—	189.5	128
6	—	—	—	—	—	—	139	93.5	64
5	—	—	—	—	—	100.5	67	45.5	32
4	—	—	—	—	71	46.5	31	21.5	16
3	—	—	—	48.25	30.5	19.5	13	9.5	8
2	—	—	30.59	17.93	10.36	6.225	4.45	4.4	4
1	—	17.23	8.875	4.819	4.275	3.651	3.176	2.6	2
0	8.050	3.868	3.860	3.340	2.825	2.342	1.856	1.4	1

TABLE 2
Probability of playing Hawk on score i/j

i/j	0	1	2	3	4	5	6	7	8
8	—	—	—	—	—	—	—	—	—
7	—	—	—	—	—	—	—	1	—
6	—	—	—	—	—	—	1	1	—
5	—	—	—	—	—	1	1	1	—
4	—	—	—	—	1	1	1	1	—
3	—	—	—	1	1	1	1	1	—
2	—	—	1	1	1	1	1	0.8	—
1	—	1	1	1	0.515	0.255	0.36	0.4	—
0	1	1	0.296	0.290	0.262	0.264	0.24	0.2	—

4.2. THE SINGLE REWARD CASE

Suppose that $C>0, V_n = V$ and $V_i = 0, i<n$.

Thus, we have the most extreme case of high reproductive skew, where a single individual receives the entire reward. Such situations occur in ant and termite colonies (Wilson, 1971). If a player loses any contest, it cannot score n/n and so the reward it receives at the end of the contest is 0.

$V_0 = \cdots = V_{n-1} = 0 \Rightarrow p_{0(n-1)} = \cdots = p_{(n-2)(n-1)} = 0$.

This in turn implies that $W_{0(n-1)} = \cdots = W_{(n-2)(n-1)} = 0$ and so on, giving $W_{ij} = 0, p_{ij} = 0$ if $i<j\leqslant n-1$. So unless a player has a 100% score, it should play Dove.

Consider W_{ii} and p_{ii}, $0\leqslant i\leqslant n-1$. $W_{i(i+1)} = 0$ so that $p_{ii} = 1$ if and only if $W_{(i+1)(i+1)}\geqslant C$.

From Result 3.2a, we know that W_{ii} is an increasing function of i, so that $p_{ii} = 1$ if and only if $i\geqslant I$ for some I. What is the value of I?

If $V<C$ then $I = n$. If $V\geqslant C$ then

$$W_{(n-1)(n-1)} = \tfrac{1}{2}(V - C).$$

If $W_{(n-1)(n-1)}\geqslant C$ then

$$W_{(n-2)(n-2)} = \tfrac{1}{2}(W_{(n-1)(n-1)} - C) = \tfrac{1}{4}V - \tfrac{3}{4}C.$$

In general, if $W_{(n-k+1)(n-k+1)}\geqslant C$, then

$$W_{(n-k)(n-k)} = \left(\frac{1}{2}\right)^{k} V - \left(1 - \left(\frac{1}{2}\right)^{k}\right) C$$

We know that $p_{II} = 1$ and $p_{(I-1)(I-1)}<1$, so that $W_{(I+1)(I+1)}\geqslant C$ but $W_{II}<C$, i.e.

$$\left(\frac{1}{2}\right)^{n-I-1} V - \left(1 - \left(\frac{1}{2}\right)^{n-I-1}\right) C\geqslant C$$

$$> \left(\frac{1}{2}\right)^{n-I} V - \left(1 - \left(\frac{1}{2}\right)^{n-I}\right) C$$

$$\Rightarrow 2^{n-I} - 1\leqslant\frac{V}{C}<2^{n-I+1} - 1$$

$$\Rightarrow n - I\leqslant\frac{\ln(1 + V/C)}{\ln 2}<n - I + 1$$

$$\Rightarrow I = n - int\left(\frac{\ln(1 + \frac{V}{C})}{\ln 2}\right).$$

Example 2. Let $n = 8, C = 5, V_8 = 256, V_i = 0$ $i<8$. The values of W_{ii} and p_{ii} are given in Table 3.

For this example $I = 3$ (i.e. $W_{ii}\geqslant 5$ if $i\geqslant 4$ and $p_{ii} = 1$ if $i\geqslant 3$).

In Example 2 there is only one reward available. If you have no chance of winning it (having already lost a contest), then play Dove. If you have a good chance of winning the reward (all games have been won with a few rounds remaining) play Hawk. If there is a small non-zero chance of winning the reward (maximum score, but early round) then play a mixed strategy.

5. Game Profiles

We define the *game profile* as the string $(r_{ij})_{i=0,\ldots,n-1;j\geqslant i}$ where $r_{ij} = 1$ if $p_{ij} = 1$ and $r_{ij} = 0$ otherwise. Thus for Example 1, the game profile is (11 000 000,1 110 000, 111 110, 11 111, 1111,111,11,1) which we can simplify in the obvious way as (64 100 000), i.e just listing the number of mixed strategies which occur for individuals with 0-wins, 1-win, and so on. This simplification is possible in general since Result 3.3 implies that for the row of scores $i/j; j = i,\ldots,n-1$, the optimal play for the earliest $I - i$ strategies is pure Hawk ($r_{ij} = 1$) and for the remaining $n - I$ strategies is a mixed strategy ($r_{ij} = 0$). This would be the order of optimal plays for a player on i/i who goes on to lose every subsequent game. The possible number of profiles for this row is thus $n - i + 1$ (I could be

TABLE 3

Expected pay-off and probability of playing Hawk on score i/i

i	0	1	2	3	4	5	6	7	8
W_{ii}	0.122	0.257	0.582	3.156	11.313	27.625	60.25	125.5	256
p_{ii}	0.051	0.116	0.631	1	1	1	1	1	—

anything from i to n). The maximum number of game profiles thus has upper bound $(n+1) \times n \times \cdots \times 2 \times 1 = (n+1)!$

Result 5.1. *For the n-round tournament, all (n+1)! profiles are possible, for every n.*

Result 5.1 is proved in the Appendix.

5.1. THE CASE WHEN $n=2$

Define

$$y = \frac{V_2 - V_1}{C} - 1, \quad x = \frac{V_1 - V_0}{C} - 1,$$

$$f(x, y) = x^2 + 3x + y, g(x, y) = x^2 + 3x - y^2 - y.$$

The game with two rounds, i.e. four players, has $3! = 6$ different profiles. Figure 1 shows the strategy space, i.e. the regions of the (x, y) plane in which each strategy is optimal (numbering is as per Table 4). For example, in the region labelled (10) [equivalent to (10,1) in the original method], the optimal play is to play Hawk in round 1, a mixed strategy in round 2 if you lose in round 1, and Hawk in round 2 if you win in round 1.

The calculations which define the regions are shown in the Appendix. These results are summarized in Table 4.

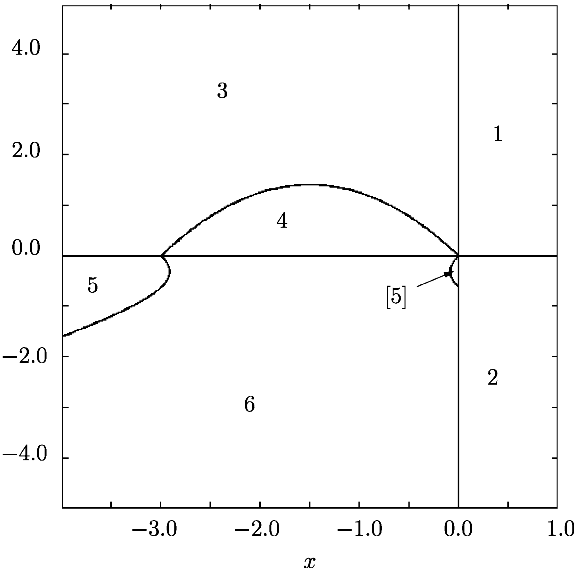

FIG. 1. Strategy space for $n = 2$ and range of x and y.

TABLE 4

Regions for the game profiles for n = 2

Profile	No.		
(00)	1	$x \geqslant 0, y \geqslant 0$	
(01)	2	$x \geqslant 0, y < 0$	
(10)	3	$x < 0, y \geqslant 0$	$f(x, y) \geqslant 0$
(20)	4	$x < 0, y \geqslant 0$	$f(x, y) < 0$
(11)	5	$x < 0, y < 0,$	$g(x, y) \geqslant 0$
(21)	6	$x < 0, y < 0,$	$g(x, y) < 0$

For example, if $V_2 = 4, V_1 = 1, V_0 = 0$ and $C = 2$, then $y = 0.5, x = -0.5$ and $f(x, y) = -0.75$ and so the optimal play is a mixed strategy in the first round, a mixed strategy in the second round if the first round has been lost, and Hawk in the second round if the first round has been won.

5.2. THE CASE WHEN $n=3$

The number of profiles is $4! = 24$.
We define the following:

$$z_1 = \frac{V_1 - V_0}{C} - 1, \quad z_2 = \frac{V_2 - V_1}{C} - 1,$$
$$z_3 = \frac{V_3 - V_2}{C} - 1.$$

The functions f and g are as defined in Section 5.1. The regions where each profile occurs are summarized in Table 5. The final two columns are new functions, $h^1_\alpha(z_1, z_2)$ and $h^2_\alpha(z_1, z_2, z_3)$ which are used in the determination of the strategy on scores $0/j$ for $j = 1, 2, 3$; α represents the strategy for non-zero scores, which of course depends only on z_2 and z_3. Given α we then find the strategy to be used on score zero via the decision tree shown in Fig. 2, where (i, α) specifies the complete strategy. Thus in Table 5, which essentially expands the information of Fig. 2, the first four rows correspond to $\alpha = (00)$, the next four to $\alpha = (01)$, and so on.

All of the conditions above are generated by using a general procedure applying to any number of rounds, which is described in the Appendix.

For example if $V_3 = 6, V_2 = 3, V_1 = 1, V_0 = 0$ and $C = 2, z_3 = 0.5, z_2 = 0, z_1 = -0.5, f(z_1, z_2) = -1.25$ and $f(0.5f(z_1, z_2), 0.5(z_2 + z_3)) = -1.234.$

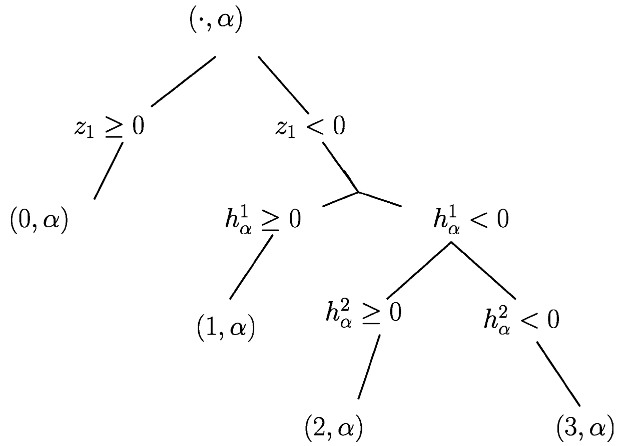

FIG. 2. Determination of first round strategy given later plays and (z_1, z_2, z_3). α is determined by (z_1, z_2, z_3).

The profile is thus (300), which means that the best play is a mixed strategy initially and whenever a player has not won a game, and Hawk at every position where at least one win has been recorded.

The three-dimensional space of (z_1, z_2, z_3) is subdivided into the 24 possible regions in a complex manner. For a slice defined by some fixed $z_1 \geqslant 0$ the regions are exactly the same shape as for $n = 2$, with α for $n = 2$ mapping into $(0, \alpha)$ for $n = 3$. For a slice defined by some fixed $z_1 < 0$ the region for a given α is (potentially) divided into pieces corresponding to $(1, \alpha)$, $(2, \alpha)$ and $(3, \alpha)$, there possibly being several pieces for a single strategy. As an illustration of this we show the subdivision of the plane defined by $z_1 = -2.98$ in which all 18 possible regions occur. The 18 regions do not always occur in such a plane, in fact the range of z_1 for which it occurs is fairly small. The total number of pieces for $z_1 = -2.98$ is 33, the profile labelled 23 being represented 5 times. Fig. 4 shows a small piece of Fig. 3 around the origin.

6. Discussion

In this paper, we have introduced a multi-player model of dominance hierarchy formation starting from a group with no established order.

TABLE 5
Regions for the game profiles for $n = 3$

Profile	No.				$h_\alpha^1(z_1, z_2)$	$h_\alpha^2(z_1, z_2, z_3)$
(000)	1	$z_2 \geqslant 0, z_3 \geqslant 0$		$z_1 \geqslant 0$		
(100)	2	$z_2 \geqslant 0, z_3 \geqslant 0$		$z_1 < 0$	$f(z_1, z_2) \geqslant 0$	
(200)	3	$z_2 \geqslant 0, z_3 \geqslant 0$		$z_1 < 0$	$f(z_1, z_2) < 0$	$f(0.5f(z_1, z_2), 0.5(z_2 + z_3)) \geqslant 0$
(300)	4	$z_2 \geqslant 0, z_3 \geqslant 0$		$z_1 < 0$	$f(z_1, z_2) < 0$	$f(0.5f(z_1, z_2), 0.5(z_2 + z_3)) < 0$
(001)	5	$z_2 \geqslant 0, z_3 < 0$		$z_1 \geqslant 0$		
(101)	6	$z_2 \geqslant 0, z_3 < 0$		$z_1 < 0$	$f(z_1, z_2) \geqslant 0$	
(201)	7	$z_2 \geqslant 0, z_3 < 0$		$z_1 < 0$	$f(z_1, z_2) < 0$	$f(0.5f(z_1, z_2), 0.5(z_2 - z_3 - z_3^2)) \geqslant 0$
(301)	8	$z_2 \geqslant 0, z_3 < 0$		$z_1 < 0$	$f(z_1, z_2) < 0$	$f(0.5f(z_1, z_2), 0.5(z_2 - z_3 - z_3^2)) < 0$
(010)	9	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) \geqslant 0$	$z_1 \geqslant 0$		
(110)	10	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) \geqslant 0$	$z_1 < 0$	$g(z_1, z_2) \geqslant 0$	
(210)	11	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) \geqslant 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$f(0.5g(z_1, z_2), 0.5(f(z_2, z_3)) \geqslant 0$
(310)	12	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) \geqslant 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$f(0.5g(z_1, z_2), 0.5(f(z_2, z_3)) < 0$
(011)	13	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) \geqslant 0$	$z_1 \geqslant 0$		
(111)	14	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) \geqslant 0$	$z_1 < 0$	$g(z_1, z_2) \geqslant 0$	
(211)	15	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) \geqslant 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$f(0.5g(z_1, z_2), 0.5(g(z_2, z_3)) \geqslant 0$
(311)	16	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) \geqslant 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$f(0.5g(z_1, z_2), 0.5(g(z_2, z_3)) < 0$
(020)	17	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) < 0$	$z_1 \geqslant 0$		
(120)	18	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) < 0$	$z_1 < 0$	$g(z_1, z_2) \geqslant 0$	
(220)	19	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) < 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$g(0.5g(z_1, z_2), 0.5(f(z_2, z_3)) \geqslant 0$
(320)	20	$z_2 < 0, z_3 \geqslant 0$	$f(z_2, z_3) < 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$g(0.5g(z_1, z_2), 0.5(f(z_2, z_3)) < 0$
(021)	21	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) < 0$	$z_1 \geqslant 0$		
(121)	22	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) < 0$	$z_1 < 0$	$g(z_1, z_2) \geqslant 0$	
(221)	23	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) < 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$g(0.5g(z_1, z_2), 0.5(g(z_2, z_3)) \geqslant 0$
(321)	24	$z_2 < 0, z_3 < 0$	$g(z_2, z_3) < 0$	$z_1 < 0$	$g(z_1, z_2) < 0$	$g(0.5g(z_1, z_2), 0.5(g(z_2, z_3)) < 0$

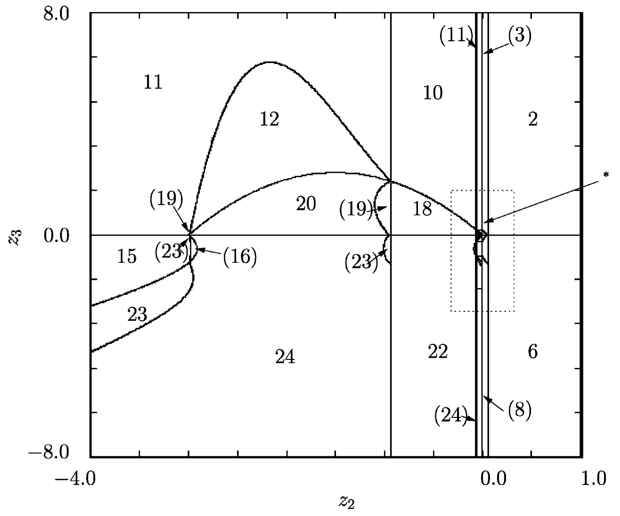

FIG. 3. Strategy space for $n = 3$ and $z_1 = -2.98$.

We further assume that all the individuals are of equal prowess and have available equivalent behavioural repetoirs. Such a group will need to establish its dominance hierarchy through a series of conflicts. There are many possible scenarios. In order to specify the particular model one needs

1. the rules of the conflict,
2. the pattern of pairing of individuals to fight (all our contests are assumed to be between a pair of animals, with no interference from others),
3. the effect of a particular outcome of a fight between two particular individuals.

We have chosen as our basic conflict the classical Hawk–Dove model, and we have further supposed that individuals are rewarded according to the score they finally obtain (number of wins) less costs; the particular individuals they have fought is no longer relevant; animal A may have beaten individual B but still be lower in the hierarchy.

The pattern of pairings is based upon a Swiss tournament. We do not claim that animals will adopt this structure exactly, but believe that under some circumstances their conflicts may approximate to it, e.g. when there is some spatial division of reward, or a geographical cue is used in mate choice. The model possesses some important features which are desirable in real populations; individuals generally fight with well-matched opponents (in the sense that they have a similar number of wins to date), the number of contests is kept to a relatively small number and a unique α-individual is produced. All these features are intrinsic to the model; they are built in. On the other hand, we have not set the behaviours or strategies across the rounds in

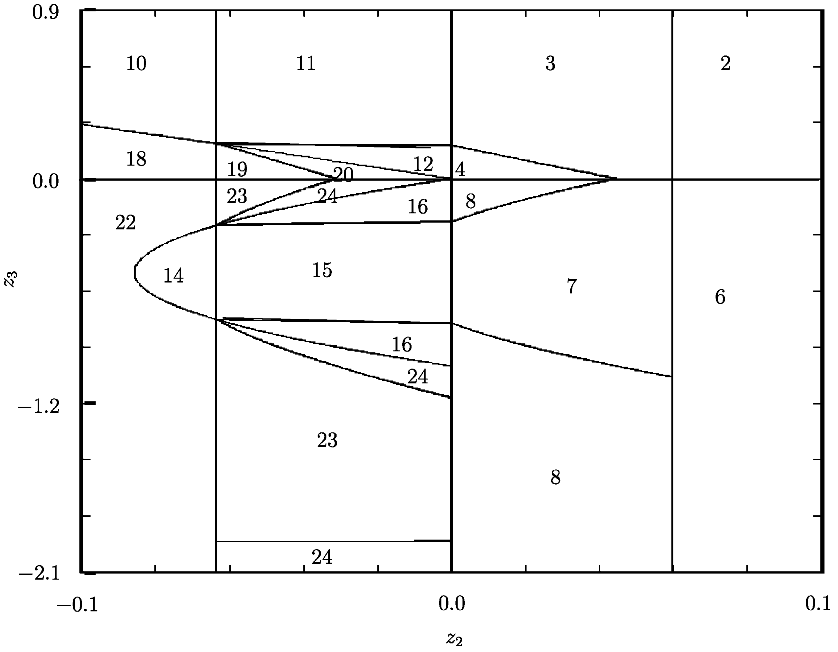

FIG. 4. Inset of strategy space for $n = 3$ and $z_1 = -2.98$.

any way. Thus, the features which we observe from the models are emergent, rather than implicit in the assumptions. We discuss some of these properties below.

In real populations the resources are often split very unevenly, so that the vast majority go to the top few individuals; Examples 1 and 2 are examples of such populations. Our model predicts that in such populations the last few contests between individuals who could become the dominant ones will be violent, whereas contests between subordinate individuals will be relatively peaceful, because at this stage there is relatively little to gain. The level of violence in the early contests will be somewhere between the two extremes, although this can be quite variable; $p_{00} = 1$ in Example 1 and $p_{00} = 0.051$ in Example 2. Note that the model described in Keller & Reeve (1994) gives conditions for when aggression is optimal in terms of a cut-off point where it is either best to be aggressive or not, whereas in our model there is a sliding scale of aggressiveness depending upon the size of the rewards available and how they are divided amongst the population.

An interesting result is Result 3.3, which implies that if the optimal strategy at any stage is a mixed strategy, then if that contest is lost, the next optimal strategy is also a mixed strategy; thus, defeats should imply a lower level of aggression for logical reasons as opposed to psychological ones, even where an animal's fighting ability is unimpaired by the defeat.

Note that some non-intuitive behaviour is possible under our model. Result 5.1 implies that all profiles allowable by Result 3.3 are possible, so that, for example, there are rewards and costs such that after three rounds individuals who have three wins employ a mixed strategy, but those with two wins and a loss play Hawk. This type of phenomenon will occur, for example, in a four round set-up if the rewards for winning three times and for winning four times are similar, and the reward for winning only twice is substantially smaller. Thus the reverse of the above result, regarding aggression following defeat, does not follow; winning a contest does not always logically imply the level of aggression should be increased.

The phenomena described above does match well with the observations on correlations in outcome. In our model there can be no such correlation but in a different context the reduction in aggression following a defeat would lead to a greater probability of losing again, while there is no necessity for a winner to increase the level of aggression. This might happen in our model if there was some inaccuracy in the pairing but individuals still adopted the strategy appropriate to their score. Reproductive skew, as typified by our numerical examples, would tend to strengthen these effects.

Finally, we expand a little on the nature of possible tournaments, and on some measures of their efficacy in determining a hierarchy. There are various ways in which one can create a multi-round contest to include the Swiss tournament as a special case. One model which includes, as its extremes, the knockout and the Swiss is the following: in each round pair individuals on the same score and after the conflicts have taken place eliminate those who have lost a specified number of conflicts, k say. If $k = 1$ then we have the knockout and if $k = n$, for an n-round contest, we have the Swiss. The ordering of the players at the end can be done through the number of contests they have won (this accounts for all the costs as well as the rewards). Furthermore, it may be the case that the particular tournament used may vary from time to time depending on the particular pattern of rewards. If there is only a single reward, all others effectively being zero, then any individual who has lost once will have nothing to gain or lose in future conflicts, and so will play Dove from then on. Effectively this means that the tournament, though organized as a Swiss is no different from a knockout. In a similar manner, one can demonstrate that if there are precisely $n+1$ non-zero payoffs then after two losses individuals play Dove, so this scenario is identical to the case $k = 2$. It is clear that further work on these aspects would be of interest.

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APPENDIX

Proof of Result 5.1. Suppose that for the $(k-1)$-round tournament, all $k!$ profiles are possible with the additional condition that $C - \varepsilon_{k-1} \leqslant V_{i+1} - V_i \leqslant C + \varepsilon_{k-1}\ \forall i$, where ε_{k-1} is small. Pick one of these profiles, and find a set of rewards which give the profile and satisfy the above condition. Consider a k-round game with rewards $V_0^*, \ldots, V_k^*$ defining $V_{j+1}^* = V_j + C\ j = 0, \ldots, k-1$ (note the C is added to ensure that there is a positive V_0^*). Let the cost again be C. After one round, the game for a player winning in the first round is identical to the $(k-1)$-round game, and so the second to kth elements of the profile are identical to the profile of the $(k-1)$-round game.

What is the optimal play in the position $0/j$ for all j? If $V_0^* = V_1^* - C - \delta$, $(\delta < \varepsilon_{k-1})$, then $p_{0(k-1)} = 1 \Rightarrow p_{0j} = 1, j < k-1$. $C - \varepsilon_{k-1} \leqslant V_{i+1}^* - V_i^* \leqslant C + \varepsilon_{k-1} \forall i$. Thus if

$V^*_{i+1} - V^*_i = C - \delta$, then

$$\begin{aligned} W_{i(k-1)} &= \frac{1}{2}(V^*_i + V^*_{i+1}) - \frac{1}{2}\left(\frac{V^*_{i+1} - V^*_i}{C}\right)^2 C \\ &= \frac{1}{2}(2V^*_i + C - \delta) - \frac{1}{2}C\left(1 - \frac{\delta}{C}\right)^2 \\ &\leqslant V^*_i + \tfrac{1}{2}\delta \leqslant V^*_i + \tfrac{1}{2}\varepsilon_{k-1}. \end{aligned}$$

If $V^*_{i+1} - V^*_i = C + \delta(\delta < \varepsilon_{k-1})$, then

$$\begin{aligned} W_{i(k-1)} &= \tfrac{1}{2}(V^*_i + V^*_{i+1}) - \tfrac{1}{2}C = V^*_i + \tfrac{1}{2}\delta \\ &\leqslant V^*_i + \tfrac{1}{2}\varepsilon_{k-1}, \end{aligned}$$

i.e. $V^*_i \leqslant W_{i(k-1)} \leqslant V^*_i + \varepsilon_{k-1}$. Thus,

$$C - 2\varepsilon_{k-1} \leqslant W_{(i+1)(k-1)} - W_{i(k-1)} \leqslant C + 2\varepsilon_{k-1}.$$

Using the same argument as above, we obtain $V^*_i \leqslant W_{i(k-1)} \leqslant W_{i(k-2)} \leqslant W_{i(k-1)} + 2\varepsilon_{k-1} \leqslant V^*_i + 4\varepsilon_{k-1}$ and

$$C - 4\varepsilon_{k-1} \leqslant W_{(i+1)(k-2)} - W_{i(k-2)} \leqslant C + 4\varepsilon_{k-1},$$

and in general, we obtain $V^*_i \leqslant W_{i(k-j)} \leqslant W_{i(k-j+1)} \leqslant W_{i(k-j+1)} + 2^{j-1}\varepsilon_{k-1} \leqslant V^*_i + 2^j\varepsilon$. Thus,

$$C - 2^j\varepsilon_{k-1} \leqslant W_{(i+1)(k-j)} - W_{i(k-j)} \leqslant C + 2^j\varepsilon_{k-1},$$

this implies that $V^*_1 \leqslant W_{11} \leqslant V^*_1 + 2^{k-1}\varepsilon_{k-1}$. We know that $W_{01} \geqslant V^*_0$, and so if we set $V^*_0 = V^*_1 - C + 2^k\varepsilon_{k-1}$, then $W_{11} - W_{01} < C$ and so $p_{00} < 1 \Rightarrow r_{00} = 0$, i.e. $r_{0j} = 0\ \forall j$.

Can we obtain all intermediate possibilities? Suppose that for some value of V^*_0, $r_{0j} = r_{0(j+1)} = 1$. Then

$$W_{0j} = \tfrac{1}{2}(W_{1(j+1)} + W_{0(j+1)} - C),$$

$$\frac{W_{1(j+1)} - W_{0(j+1)}}{C} - 1 \geqslant 0,$$

$$\begin{aligned} &\frac{W_{1j} - W_{0j}}{C} - 1 \\ &= \frac{W_{1j} - \frac{1}{2}W_{1(j+1)} - \frac{1}{2}W_{0(j+1)} + \frac{1}{2}C}{C} - 1 \\ &= \frac{1}{2}\left(\frac{W_{1(j+1)} - W_{0(j+1)}}{C} - 1\right) \\ &\quad + \frac{W_{1j} - W_{1(j+1)}}{C} \geqslant 0. \end{aligned}$$

Note that $W_{1j} - W_{1(j+1)} = 0$ if and only if $W_{2(j+1)} = W_{1(j+1)}$ or $W_{2(j+1)} = W_{1(j+1)} + C$. This occurs on a region of dimension less than the parameter space, and so it is easy to find a set of $V_i s$ with the required properties where this does not occur.

Thus as we increase V^*_0, the first of these inequalities will be violated before the second. Thus as V^*_0 increases, the number of r_{0j}'s which are equal to 0 increases by 1 at certain points. Both extreme values of this number (0 and $k+1$) are possible, so that all values are possible, with the condition $C - \varepsilon_k < V^*_{j+1} - V^*_j \leqslant C + \varepsilon_k$ satisfied, where $\varepsilon_k = 2^k\varepsilon_{k-1}$.

We can repeat this argument for any of the profiles of the $(k-1)$-round game, thus giving all $k!$ profiles of the k-round game, with the rewards satisfying the same condition. Both profiles are possible for the 1-round game. Thus, by induction, all $(n+1)!$ profiles are possible for the n-round game for all n. □

Calculations defining the regions for $n = 2$. We shall find the values of r_{00}, r_{01} and r_{11} under all possible values of x and y.

$y \geqslant 0 \Rightarrow V_2 - V_1 \geqslant C \Rightarrow r_{11} = 1$.
If $y < 0$ then $r_{11} = 0$.
$x \geqslant 0 \Rightarrow r_{01} = 1 \Rightarrow r_{00} = 1$.
If $x < 0$ then $r_{01} = 0$.

So if $x \geqslant 0, y \geqslant 0$ the profile is (00), if $x \geqslant 0, y < 0$ the profile is (01).

If $x < 0$ and $y \geqslant 0$ then $r_{01} = 0, r_{11} = 1$. $r_{00} = 1$ if $W_{11} - W_{01} \geqslant C$, i.e.

$$\begin{aligned} &\frac{1}{2}(V_2 + V_1 - C) \\ &\quad - \frac{1}{2}\left(V_1 + V_0 - \frac{1}{C}(V_1 - V_0)^2\right) \geqslant C \\ &\quad \Rightarrow C(V_2 - V_1) + C(V_1 - V_0) \\ &\quad + (V_1 - V_0)^2 - 3C^2 \geqslant 0 \\ &\quad \Rightarrow f(x, y) = x^2 + 3x + y \geqslant 0. \end{aligned}$$

Otherwise $r_{00} = 0$.

So if $x < 0, y \geqslant 0, f(x, y) \geqslant 0$ the profile is (10), if $x < 0, y \geqslant 0, f(x, y) < 0$ the profile is (20).

If $x<0$ and $y<0$ then $r_{01}=0, r_{11}=0$. $r_{00}=1$ if $W_{11}-W_{01}\geqslant C$, i.e.

$$\frac{1}{2}\left(V_2+V_1-\frac{1}{C}(V_2-V_1)^2\right)$$
$$-\frac{1}{2}\left(V_1+V_0-\frac{1}{C}(V_1-V_0)^2\right)\geqslant C$$
$$\Rightarrow C(V_2-V_1)+C(V_1-V_0)-(V_2-V_1)^2$$
$$+(V_1-V_0)^2-2C^2\geqslant 0$$
$$\Rightarrow g(x,y)=x^2+3x-y^2-y\geqslant 0.$$

Otherwise $r_{00}=0$.

So if $x<0, y<0, g(x,y)\geqslant 0$ the profile is (11), if $x<0, y<0, g(x,y)<0$ the profile is (21).

A general procedure for finding game profiles. Define z_{ij} by

$$z_{ij}=\frac{W_{(i+1)(j+1)}-W_{i(j+1)}}{C}-1.$$

Thus $r_{ij}=1$ if and only if $z_{ij}\geqslant 0$. We will find z_{ij} as a function of $z_{(i+1)(j+1)}$ and $z_{i(j+1)}$. There are four cases to consider.

(1) $r_{(i+1)(j+1)}=1, r_{i(j+1)}=1$. Thus the optimal play on the scores $i+1/j+1$ and $i/j+1$ is to play Hawk. This is equivalent to $z_{(i+1)(j+1)}\geqslant 0, z_{i(j+1)}\geqslant 0$.

$$W_{(i+1)(j+1)}-W_{i(j+1)}=\tfrac{1}{2}(W_{(i+2)(j+2)}$$
$$-W_{(i+1)(j+2)})+\tfrac{1}{2}(W_{(i+1)(j+2)}-W_{i(j+2)})$$
$$\Rightarrow z_{ij}=\tfrac{1}{2}(z_{(i+1)(j+1)}+z_{i(j+1)}).$$

Note that z_{ij} must be positive, as Result 3.3 demands.

(2) $r_{(i+1)(j+1)}=1, r_{i(j+1)}=0$. The optimal play on score $(i+1)/(j+1)$ is to play Hawk, the optimal play on $i/(j+1)$ is a mixed strategy. Thus, $z_{(i+1)(j+1)}\geqslant 0, z_{i(j+1)}<0$.

$$W_{(i+1)(j+1)}-W_{i(j+1)}=\frac{1}{2}(W_{(i+2)(j+2)}$$
$$-W_{(i+1)(j+2)})+\frac{1}{2}(W_{(i+1)(j+2)}-W_{i(j+2)})$$
$$-\frac{1}{2}C+\frac{1}{2C}(W_{(i+1)(j+2)}-W_{i(j+2)})^2$$
$$\Rightarrow z_{ij}=\frac{1}{2}(3z_{i(j+1)}+z_{(i+1)(j+1)}+z^2_{i(j+1)})$$
$$=\frac{1}{2}f(z_{i(j+1)},z_{(i+1)(j+1)}).$$

z_{ij} may be positive or negative, and so r_{ij} can be 1 or 0.

(3) $r_{(i+1)(j+1)}=0, r_{i(j+1)}=1$. The optimal play on score $(i+1)/(j+1)$ is to play a mixed strategy, the optimal play on $i/(j+1)$ is a to play Hawk. Thus, $z_{(i+1)(j+1)}<0, z_{i(j+1)}\geqslant 0$.

$$W_{(i+1)(j+1)}-W_{i(j+1)}=\frac{1}{2}(W_{(i+2)(j+2)}-W_{(i+1)(j+2)})$$
$$+\frac{1}{2}(W_{(i+1)(j+2)}-W_{i(j+2)})$$
$$+\frac{1}{2}C-\frac{1}{2C}(W_{(i+2)(j+2)}-W_{(i+1)(j+2)})^2$$
$$\Rightarrow z_{ij}=\frac{1}{2}(z_{i(j+1)}-z_{(i+1)(j+1)}-z^2_{(i+1)(j+1)}).$$

It is again clear that z_{ij} is positive and so $r_{ij}=1$, as implied by Result 3.3.

(4) $r_{(i+1)(j+1)}=0, r_{i(j+1)}=0$. The optimal play on the scores $i+1/j+1$ and $i/j+1$ is a mixed strategy, i.e. $z_{(i+1)(j+1)}<0, z_{i(j+1)}<0$.

$$W_{(i+1)(j+1)}-W_{i(j+1)}=\frac{1}{2}(W_{(i+2)(j+2)}-W_{(i+1)(j+2)})$$
$$+\frac{1}{2}(W_{(i+1)(j+2)}-W_{i(j+2)})$$
$$-\frac{1}{2C}(W_{(i+2)(j+2)}-W_{(i+1)(j+2)})^2$$
$$+\frac{1}{2C}(W_{(i+1)(j+2)}-W_{i(j+2)})^2$$
$$\Rightarrow z_{ij}=\frac{1}{2}(3z_{i(j+1)}-z_{(i+1)(j+1)}+z^2_{i(j+1)}-z^2_{(i+1)(j+1)})$$
$$=\frac{1}{2}g(z_{i(j+1)},z_{(i+1)(j+1)}).$$

z_{ij} may be positive or negative, and so r_{ij} can be 1 or 0.

From the above four cases, a general procedure can be constructed to find the values of z_{ij} and r_{ij} for every pair i,j $i\leqslant j$. Thus, the profile of an arbitrarily large knockout game can be found. Moreover, the values of p_{ij} and W_{ij} are found along the way, due to the relationship

$$p_{ij}=1+min(z_{ij},0)$$

and eqn (1).