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Sequential methods for generating patterns of ESS's

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Abstract. A finite conflict with given payoff matrix may have many ESS's (evolutionarily stable strategies). For a given set of pure strategies $\{1, 2, \ldots, n\}$ a set of subsets of these is called a pattern, and if there exists an $n \times n$ matrix which has ESS's whose supports (i.e. the playable strategies) precisely match the elements of the pattern, then the pattern is said to be attainable. In [5] and [10] some methods were developed to specify when a pattern was, or was not, attainable. The object here is to present a somewhat different method which is essentially recursive. We derive certain results which allow one to deduce from the attainability of a pattern for given n the attainability of other patterns for $n+1$, and by induction for any $n+r$.

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1 Introduction

The notion of an Evolutionarily Stable Strategy (which is defined below) has become of major importance in studying the strategies adopted by organisms. An ESS corresponds to a strategy, which if adopted by a population in some conflict, cannot be invaded by any alternative introduced at low frequency. Such a strategy is therefore stable; it will persist if current payoffs and costs remain the same, and if no new pure strategies become available.

An interesting feature of conflicts with a finite number of available pure strategies, where the payoffs can most naturally be specified by a (payoff) matrix, is the possible existence of multiple ESS's. The ESS's which exist will have supports which are restricted in various ways, and the discussion of these restrictions is the subject of a series of papers by the second and third of the current authors, and various collaborators [10, 5]. Biological relevence lies in the possibility that for a particular conflict with a specific payoff matrix one may observe, in separated niches, different strategy combinations depending on which ESS's have evolved in those niches. Conversely, observation of a particular pattern of supports may imply that the niches do not all have the same payoff matrix. Additionally it is of interest to know how many different ESS's might exist for a given number of pure strategies, and how large these ESS's might be.

The current paper presents new methods for addressing questions regarding possible patterns of ESS's.

2 Evolutionarily stable strategies

Suppose that pairwise contests are taking place within a species, each player choosing from a set $\boldsymbol{U}$ of pure strategies, with pay-offs a_{ij} forming an $n \times n$ matrix, $\boldsymbol{A}$. The concept of an ESS, corresponding to a non-invadable population strategy, of $\boldsymbol{A}$ was introduced by Maynard Smith and Price [9]. The expected pay-off to an i-player when it meets a j-player is $E[i,j]=a_{ij}$. Let the proportion of the individuals in the population which play i be p_i (some of these may be zero). The pay-off to a group of individuals of which a proportion p_i play i, against a group of which a proportion q_i play i is

$$E[\boldsymbol{p},\boldsymbol{q}]=\sum_{ij} p_i q_j E[i,j] .$$

$\boldsymbol{p}$ is said to be ES (evolutionarily stable) against $\boldsymbol{q}$ w.r.t. $\boldsymbol{A}$ if

(i) $E[\boldsymbol{p},\boldsymbol{p}]>E[\boldsymbol{q},\boldsymbol{p}]$ or

(ii) $E[\boldsymbol{p},\boldsymbol{p}]=E[\boldsymbol{q},\boldsymbol{p}]$ and $E[\boldsymbol{p},\boldsymbol{q}]>E[\boldsymbol{q},\boldsymbol{q}]$.

$\boldsymbol{p}$ is said to be an ESS (evolutionarily stable strategy) of $\boldsymbol{A}$ if for all $\boldsymbol{q} \neq \boldsymbol{p}$ $\boldsymbol{p}$ is ES against $\boldsymbol{q}$ w.r.t $\boldsymbol{A}$.

The *support* of an ESS $\boldsymbol{p}$ is $\boldsymbol{S}(\boldsymbol{p})=\{i;\, i\in\boldsymbol{U},\ p_i>0\}$, i.e. it is the set of strategies which have a non-zero probability of being played by an individual who plays the ESS $\boldsymbol{p}$. We shall drop the $\boldsymbol{p}$ from $\boldsymbol{S}(\boldsymbol{p})$ where no ambiguity will result.

2.1 Haigh's theorem

Conditions for a vector $\boldsymbol{p}$ to be an ESS of a payoff matrix $\boldsymbol{A}$ were given by Haigh [8]. In the generic case we require

(1) $E[i,\boldsymbol{p}]=c$ for $i\in\boldsymbol{S}(\boldsymbol{p})$ where c is constant (such a $\boldsymbol{p}$ is said to be *an equilibrium* over $\boldsymbol{S}(\boldsymbol{p})$), and $c>E[j,\boldsymbol{p}]$ for $j\in\boldsymbol{U}\setminus\boldsymbol{S}(\boldsymbol{p})$, that is, j cannot *invade* $\boldsymbol{p}$.

(2) If $\boldsymbol{B}$ is defined by $b_{ij}=a_{ij}$ for $i,j\in\boldsymbol{S}(\boldsymbol{p})$ and $\boldsymbol{p}^*$ by $p_i^*=p_i$ for $i\in S(\boldsymbol{p})$, so that $\boldsymbol{p}^*=(\boldsymbol{B}^{-1}.\mathbf{1})/(\mathbf{1}^{\mathrm{T}}.\boldsymbol{B}^{-1}.\mathbf{1})$, then if $\boldsymbol{C}$ is defined by $c_{ij}=(b_{ij}-b_{ik}-b_{kj}+b_{kk})$ for $k\in\boldsymbol{S}(\boldsymbol{p})$ fixed and $i,j\in\boldsymbol{S}(\boldsymbol{p})\setminus\{k\}$, we require that $\boldsymbol{C}$ is negative definite. The negative definite requirement on $\boldsymbol{C}$ arises in Haigh's theorem through a requirement that $\boldsymbol{z}^{\mathrm{T}}(\boldsymbol{B}+\boldsymbol{B}^{\mathrm{T}})\boldsymbol{z}<0$ for all non-zero $\boldsymbol{z}$ whose elements sum to zero. We shall refer to this condition on $\boldsymbol{B}$ as "the negative definiteness condition" where no confusion will result.

This pair of conditions is necessary and sufficient in the generic case (but see [1] for a discussion of the non-generic case).

Another useful notion is that of *domination*; the ith-row of the matrix A is said to dominate the jth row if

(1) $a_{ik} \geqq a_{jk} \forall k$
(2) $\exists l$ such that $a_{il} > a_{jl}$.

3 Patterns of attainable ESS's

Any set of supports $\{S_1, S_2, \ldots, S_k\}$ is called a *pattern*. If we have a specific pattern in mind, e.g. the supports are $\{1,2,3\}$, $\{1,4\}$ and $\{2,4\}$, the pattern is written as $\{(123)(14)(24)\}$.

A pattern is said to be *attainable* if and only if there is a payoff matrix A with ESS's whose supports form that pattern. We shall also say that a (specific) pattern is *attained* by a given matrix.

A pattern is said to be *maximal* if it is attainable and it is not a proper subset of another attainable pattern.

A pattern $\{T_1, \ldots, T_k\}$ is said to be *degenerate* if $\{1, \ldots, k\}$ can be partitioned into two non-empty set X, Y s.t. $(\bigcup_{i \in X} T_i) \cap (\bigcup_{i \in Y} T_i) = \varnothing$

The following theorem is fundamental to the study of patterns of ESS's, and is taken as read throughout this paper.

3.1

Bishop and Canning's theorem *If I and J are the supports of ESS's of some payoff matrix A, then neither I nor J is a subset of the other* [2].

There are now several papers dealing with the subject of patterns of ESS's. The only known method of showing that a particular pattern is attainable is by finding a payoff matrix which has this pattern. Conversely a pattern is shown not to be attainable if it can be proved that no such matrix exists. So, for example, [10] gives several theorems which show that patterns of particular types do not exist and [5] shows how to construct matrices of a particular type generating a special class of patterns.

Cannings and Vickers [10, 5] conjecture that if a pattern is attainable then so is any subset of that pattern. If this is correct then a complete specification of the set of attainable patterns is implicit in a list of the maximal, non-degenerate, attainable patterns.

Cannings and Vickers [7] gave details of the progress made in the attempt to specify all the maximal, non-degenerate attainable patterns in the 5-strategy case (the 4-strategy case being complete). There are a small number of patterns whose attainability is unknown (i.e. no payoff matrix has been found with that pattern, but it has not been proven that such a matrix cannot exist). Many potential patterns have been shown not to exist by use of exclusion theorems (principally those in [10]). Out of the patterns which have been shown to be attainable (in particular the maximal patterns) most have been found by individual construction or computer generated trial and error. The only methods so far available for generating patterns are the clique matrices of [5], which cannot give all the patterns in the 4-strategy case, and which find only five out of the sixteen known maximal patterns in the 5-strategy case, (although they do generate all attainable patterns with supports of size three or less) and methods for special types of pattern (e.g. one n-element ESS plus pair ESS's see [3]).

4 Sequential methods

This paper provides more methods for generating general patterns; in particular showing the existence of patterns on n strategies conditional on the existence of patterns on $n-1$ strategies. These methods generate all the patterns known attainable in [7] for the 5-strategy case except $\{(123)(234)(345)(451)(512)\}$. Some further progress has been made for $n=5$ which will be presented elsewhere.

Given that we know that a certain pattern $\{S_1, \ldots, S_k\}$ exists on strategies $\{1, \ldots, n-1\}$, can we say that certain patterns exist on $\{1, \ldots, n\}$? Trivially we can say that $\{S_1, \ldots, S_k\}$ and $\{S_1, \ldots, S_k, (n)\}$ are attainable, but what of more interesting patterns? For all-but-one of the theorems in this paper, the $(n-1)\times(n-1)$ payoff matrix A which gives the pattern $\{S_1, \ldots, S_k\}$ on $\{1, \ldots, n-1\}$ will be kept fixed and only the elements of the nth row and column introduced by the addition of a new strategy n will be varied.

There are two basic methods used in this paper, *splitting* and *adding*. Splitting is so-called because one (or more) elements is (are) 'split' into two, i.e. if the pattern is $\{(12)\}$ and 1 is split, then the new pattern is $\{(12)(32)\}$. This is the method used in Theorem 1. Theorem 1 splits one element, say 1, so that every support which contains a 1 is copied and a new support is formed with 1 replaced by the new strategy, e.g. $\{(123)(14)(24)\}\to\{(123)(523)(14)(54)(24)\}$; 1 has been 'split' into 1 and 5. Theorem 3 uses a more complicated version of this basic method, splitting two strategies simultaneously.

Adding is used in Theorems 2, 4, 5, 6, and occurs when a new strategy is 'added' to a support; e.g. if the pattern is $\{(12)\}$, 3 can be added to make the pattern $\{(123)\}$. This is all that happens in Theorem 2. Part a) adds the new element to every support, so that, for example, $\{(123)(14)(24)\}\to\{(1235)(145)(245)\}$ and parts b)–d) add the new element only to some subset of the supports. Theorems 4–6 are more complicated. They create new ESS's and do not seem to be adding in the same way as in Theorem 2, but in reality the method is very similar. They create new supports which include the new strategy, 'adding' this strategy to sets of strategies which, although they did not form the supports of ESS's, satisfied enough of the conditions to make the new supports attainable.

Some of these methods are also of use when considering the more restricted case of symmetric payoff matrices, and are used in [4]. In particular Theorems 1, 2a), 2b), 3, 5a), 5b) still apply. Theorem 2c) still works for $i=1$, although a proof with a form similar to that of Theorem 1 is required to show this.

In this paper every payoff matrix is assumed to have zeros down the leading diagonal, i.e. $a_{ii}=0$ for all i. That it is only necessary to consider such matrices was first pointed out by Zeeman [11]. Such matrices are said to be in 'reduced form'. All the theorems in this paper are, obviously, true if the indices are permuted.

4.1

Theorem 1 *If the pattern* $\{S_1, \ldots, S_t, S_{t+1}, \ldots, S_k\}$ *is attainable on* $\{1, \ldots, n-1\}$ *and such that* $1\in S_i$ *iff* $i\leq t$, *then* $\{S_1, \ldots, S_t, S'_1, \ldots, S'_t, S_{t+1}, \ldots, S_k\}$ *is attainable on* $\{1, \ldots, n\}$ *where* $S'_i=\{n\}\cup S_i\setminus\{1\}$.

Proof. Let $A=(a_{ij})$, $i,j=1,\ldots,n-1$ be a matrix with pattern $\{S_1,\ldots,S_t, S_{t+1},\ldots,S_k\}$. Now let $A'=(a'_{ij})$ be an $n\times n$ matrix s.t.

$$\begin{aligned}
a'_{ij}&=a_{ij} && i,j\leqq n-1\\
a'_{nj}&=a_{1j} && 2\leqq j\leqq n-1\\
a'_{jn}&=a_{j1} && 2\leqq j\leqq n-1\\
a'_{nn}&=0 && a'_{1n}=a'_{n1}=-1\,.
\end{aligned}$$

With column n removed, row 1 of A' dominates row n, so that any ESS with support containing 1 cannot be invaded by n and any ESS with support not containing 1 cannot be invaded by n since it cannot be invaded by 1. So any ESS on A is an ESS of A'. Consider A^+, the submatrix of A' with the first row and column removed but then A^+ is just the same matrix as A with subscript 1 replaced by n. So for every ESS of A there is an ESS of A^+ with 1 replaced by n in its support. Omitting column 1, row n dominates row 1, so that every ESS on A^+ is an ESS of A'.

No support of an ESS of A' can contain both 1 and n, since the negative definiteness condition would be violated (all pairs a_{ij} and a_{ji} s.t. i and j are in the support of the ESS must sum to a positive value). So if any other ESS's exist they must be of A^+, since we know the pattern of A. But using a similar argument to that used above, this would mean that the corresponding ESS with strategy 1 instead of strategy n would be an ESS of A (if a support does not include an n, then it is already the support of an ESS of A). But such an ESS does not exist, so the only 'new' ESS's are those described above. Thus Theorem 1 is proved.

An example of Theorem 1 is as follows:
The pattern $\{(123)(24)(34)\}$ is attained by the payoff matrix

$$\boldsymbol{M1}=\begin{bmatrix} 0 & -1 & 2 & -3\\ 2 & 0 & -1 & 3\\ -1 & 2 & 0 & 2\\ -8 & 7 & 1 & 0\end{bmatrix}.$$

Theorem 1 shows that the pattern $\{(123)(125)(24)(34)(54)\}$ is attainable (here 3 has been split into 3 and 5 rather than splitting index 1 as in the statement of the theorem). The matrix $\boldsymbol{M2}$ is a payoff matrix for this pattern, obtained from the construction for Theorem 1, where

$$\boldsymbol{M2}=\begin{bmatrix} 0 & -1 & 2 & -3 & 2\\ 2 & 0 & -1 & 3 & -1\\ -1 & 2 & 0 & 2 & -1\\ -8 & 7 & 1 & 0 & 1\\ -1 & 2 & -1 & 2 & 0\end{bmatrix}.$$

4.2

Theorem 2 *Suppose that the pattern* $\{S_1, \ldots, S_k\}$ *is attainable on the strategies* $\{1, \ldots, n-1\}$. *Then the pattern* $\{S'_1, \ldots, S'_k\}$ *is attainable on* $\{1, \ldots, n\}$ *in each of the following cases;*

a) $S'_j = S_j \cup \{n\}$, *all* j.
b) $S'_j = S_j \cup \{n\}$, *if none of* $1, \ldots, i$ *are in* S_j, $i < n-1$, *otherwise* $S'_j = S_j$.
c) $S'_j = S_j \cup \{n\}$, *if* S_j *contains at least one of* $1, \ldots, i$, $i < n-1$, *otherwise* $S'_j = S_j$.
d) *Partition* $\{1, \ldots, n-1\}$ *into non-overlapping sets* $U_1, \ldots, U_s$

(i.e. each i is contained in one and only one of the U_j*'s). Then*

$S'_j = S_j \cup \{n\}$ *if* $S_j \subset \bigcup_{l=1}^{2t+1} U_l$ *but* $S_j \not\subset \bigcup_{l=1}^{2t} U_l$ *some* $t \leqq (s-1)/2$, *otherwise* $S'_j = S_j$.

a), b) and c) are special cases of d), with U_l*'s* defined as follows;

a) $s=1$, $U_1 = \{1, \ldots, n-1\}$
b) $s=2$, $U_1 = \{i+1, \ldots, n-1\}$, $U_2 = \{1, \ldots, i\}$
c) $s=3$, $U_1 = \varnothing$, $U_2 = \{i+1, \ldots, n-1\}$, $U_3 = \{1, \ldots, i\}$.

The construction of the new payoff matrix A' for Theorem 2 is given as follows: For each of a)–d)

$$a'_{ij} = a_{ij} \quad i, j = 1, \ldots, n-1 \qquad \text{and} \qquad a_{nn} = 0$$

a) $a'_{nj} = M$, $a'_{jn} = 1$ $j = 1, \ldots, n-1$ where $M = \sup\{a_{ij}\}$ (or $M = 1$ if $n = 2$).
b) $a'_{nj} = M$ $j = i+1, \ldots, n-1$: $a'_{nj} = -\alpha M$ $j = 1, \ldots, i$: $a'_{jn} = 1$ $j = 1, \ldots, n-1$.
c) $a'_{nj} = \alpha M$ $j = 1, \ldots, i$: $a'_{nj} = 0$ $j = i+1, \ldots, n-1$: $a'_{jn} = \beta$ $j = 1, \ldots, n-1$.
d) $a'_{nj} = (-2\alpha)^{l-1} M$ where l is such that $j \in U_l$ $\forall j \in \{1, 2, \ldots, n-1\}$: $a'_{jn} = \beta$ $j = 1, \ldots, n-1$

where $\alpha = 1/(\inf p_{st})$, p_{st} being the tth non-zero element in the sth ESS of A, and β is sufficiently large. Theorem 2 is proved in Sect. 5.1.

For example, the pattern $\{(123)(24)(34)\}$ is attained by the payoff matrix

$$M3 = \begin{bmatrix} 0 & -1 & 2 & -3 \\ 2 & 0 & -1 & 3 \\ -1 & 2 & 0 & 2 \\ -8 & 7 & 1 & 0 \end{bmatrix}.$$

Using Theorem 2 part a) we get the pattern $\{(1235)(245)(345)\}$, attained by the matrix

$$M4 = \begin{bmatrix} 0 & -1 & 2 & -3 & 1 \\ 2 & 0 & -1 & 3 & 1 \\ -1 & 2 & 0 & 2 & 1 \\ -8 & 7 & 1 & 0 & 1 \\ 7 & 7 & 7 & 7 & 0 \end{bmatrix}.$$

Theorem 2b) with $\{1, 3\}$ as $\{1, \ldots, i\}$ shows that the pattern $\{(123)(245)(34)\}$ is attained by the matrix

$$M5 = \begin{bmatrix} 0 & -1 & 2 & -3 & 1 \\ 2 & 0 & -1 & 3 & 1 \\ -1 & 2 & 0 & 2 & 1 \\ -8 & 7 & 1 & 0 & 1 \\ -70/3 & 7 & -70/3 & 7 & 0 \end{bmatrix}.$$

Part c), again with $\{1, 3\}$ as $\{1, \ldots, i\}$ generates the pattern $\{(1235)(24)(345)\}$, which is attained by the payoff matrix

$$M6 = \begin{bmatrix} 0 & -1 & 2 & -3 & 500 \\ 2 & 0 & -1 & 3 & 500 \\ -1 & 2 & 0 & 2 & 500 \\ -8 & 7 & 1 & 0 & 500 \\ 70/3 & 0 & 70/3 & 0 & 0 \end{bmatrix}.$$

There are two other special cases of Theorem 2 which are of interest:

1) $U_1 = S_1$, $U_2 = \{1, \ldots, n-1\} \setminus S_1$ adds n to S_1 but leaves all other supports unchanged.

2) $U_1 = \varnothing$, $U_2 = S_1$, $U_3 = \{1, \ldots, n-1\} \setminus S_1$ adds n to all supports except S_1, which remains unchanged.

In fact the entire theorem is a special case of Theorem 5, as will be seen later.

Patterns on larger numbers of strategies can be built up from those on smaller numbers as follows:

$\{(1)\}$ is attained by payoff matrix [0].
$\{(1)(2)\}$ is attained by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

using Theorem 1.

Then using Theorem 2a), $\{(13)(23)\}$ is attained by the matrix

$$M7 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

From this matrix we can generate the pattern $\{(13)(23)(34)\}$, attained by

$$M8 = \begin{bmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

and $\{(13)(23)(14)(24)\}$, attained by

$$\boldsymbol{M9}=\begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

using the construction for Theorem 1. From $\boldsymbol{M9}$, using Theorem 2a), we can construct a matrix with the pattern $\{(135)(235)(145)(245)\}$, viz.

$$\boldsymbol{M10}=\begin{bmatrix} 0 & -1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

In this way patterns on larger numbers of strategies can be built up.

In [5] a class of matrices was defined which yielded many useful results. For this class matrices were symmetric, had 0's on the diagonal, and $+1$ or -1 off the diagonal. The ESS's of such a matrix have supports matching the (maximal) cliques of the graph whose adjacency matrix is obtained from the payoff matrix by substituting 0 for each -1. We refer to such payoff matrices as *clique matrices*, though this term is used differently in the general graph theory literature, and resulting pattern as *clique patterns*.

The matrices given above are all clique matrices, and using Theorem 1 or 2a) (as well as 2b) or 3) clique patterns will only generate clique patterns. However, using the other theorems non-clique patterns can be generated from clique patterns. The methods in this paper are of greater interest when applied to patterns which cannot be attained by clique matrices but which have been found using other methods. For example the pattern $\{(123)(345)(14)(15)\}$ exists for the 5-strategy case and is attained by the following payoff matrix

$$\boldsymbol{M11}=\begin{bmatrix} 0 & 96 & -95 & 35 & 75 \\ -95 & 0 & 96 & -500 & -500 \\ 96 & -95 & 0 & 8 & -6 \\ 150 & -500 & -6 & 0 & 24 \\ 225 & -500 & 8 & -18 & 0 \end{bmatrix}.$$

Now using Theorem 1 (copying ESS's with 1 to ESS's with a 6 instead of a 1), we get the pattern $\{(123)(345)(236)(14)(15)(46)(56)\}$ on 6 strategies. The matrix generated is

$$\boldsymbol{M12}=\begin{bmatrix} 0 & 96 & -95 & 35 & 75 & -1 \\ -95 & 0 & 96 & -500 & -500 & -95 \\ 96 & -95 & 0 & 8 & -6 & 96 \\ 150 & -500 & -6 & 0 & 24 & 150 \\ 225 & -500 & 8 & -18 & 0 & 225 \\ -1 & 96 & -95 & 35 & 75 & 0 \end{bmatrix}.$$

Similarly starting from the same original payoff matrix, using Theorem 2a), we get the pattern $\{(1236)(3456)(146)(156)\}$. The matrix generated is

$$\boldsymbol{M}\,\mathbf{13}=\begin{bmatrix} 0 & 96 & -95 & 35 & 75 & 1 \\ -95 & 0 & 96 & -500 & -500 & 1 \\ 96 & -95 & 0 & 8 & -6 & 1 \\ 150 & -500 & -6 & 0 & 24 & 1 \\ 225 & -500 & 8 & -18 & 0 & 1 \\ 96 & 96 & 96 & 96 & 96 & 0 \end{bmatrix}.$$

Theorems 1 and 2 (here only 2a)) have been considered here because they are both very useful and fairly simple to follow. From the above examples it can be seen that complex patterns can be generated by these theorems very easily. On the other hand Theorem 3 is of only marginal importance, while Theorems 4–6 involve extra conditions than just knowing a particular pattern exists on $n-1$ strategies.

4.3

Theorem 3 *Suppose that the pattern* $\{S_1, \ldots, S_t, S_{t+1}, \ldots, S_k\}$ *is attainable on* $n-2$ *strategies where,* $\{1, 2\} \subset S_i$ iff $i \leqq t$. *Then the pattern,*

$$\{S_1, \ldots, S_t, S_1^1, \ldots, S_t^1, S_1^2, \ldots, S_t^2, S_{t+1}, \ldots, S_k, (n-1, n)\}$$

is attainable, where $S_i^1 = S_i \cup \{n-1\} \setminus \{1\}$ *and* $S_i^2 = S_i \cup \{n\} \setminus \{2\}$ *for* $1 \leqq i \leqq t$.

The construction of the new payoff matrix, together with a proof of the theorem, is given in Sect. 6.2.

As an example the pattern $\{(123)\}$ on 3 strategies is attained by the matrix

$$\boldsymbol{M}\,\mathbf{14}=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and using the construction for Theorem 3, we see that the pattern $\{(123)(234)(135)(45)\}$ is attained by the payoff matrix

$$\boldsymbol{M}\,\mathbf{15}=\begin{bmatrix} 0 & 1 & 1 & -36 & 1 \\ 1 & 0 & 1 & 1 & 36 \\ 1 & 1 & 0 & 1 & 1 \\ -36 & 1 & 1 & 0 & 6 \\ 1 & -36 & 1 & 6 & 0 \end{bmatrix}.$$

The next theorem refers to negative definiteness and we remind the reader that a matrix $\boldsymbol{B}$ is said to satisfy the negative definiteness condition if and only if $z^{\mathrm{T}}(\boldsymbol{B}+\boldsymbol{B}^{\mathrm{T}})z<0$ for every non-zero vector z whose elements sum to zero.

4.4

Theorem 4 *Suppose the pattern* $\{S_1, \ldots, S_k\}$ *is attainable on* $\{1, \ldots, n-1\}$ *and that* I *is a subset of* $\{1, \ldots, n-1\}$ *for which* $S_i \not\subset I$, $i=1, \ldots, k$. *Suppose further that* A_I, *the submatrix of* A *corresponding to the set* I, *satisfies the negative definiteness condition. Then the pattern* $\{S_1, \ldots, S_k, I'\}$ *is attainable on* $\{1, \ldots, n\}$ *where* I' *is the set* $I \cup \{n\}$.

The proof of the theorem, which includes the construction for the matrix, is given in Sect. 6.3.

For example, a matrix with attains the pattern $\{(12)(23)\}$ is

$$\boldsymbol{M16} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 7 \\ -2 & 2 & 0 \end{bmatrix}.$$

Also $\begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}$ satisfies the negative definiteness condition (which for 2 by 2 matrices is $a_{12}+a_{21}-a_{11}-a_{22}>0$), and so the matrix

$$\boldsymbol{M17} = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 7 & -35 \\ -2 & 2 & 0 & 1.5 \\ 6.75 & -35 & 6.25 & 0 \end{bmatrix}$$

attains the pattern $\{(12)(23)(134)\}$ which cannot be achieved by cliques.

4.5

Theorem 5 *Suppose that* $\{T_1, \ldots, T_j, T_{j+1}, \ldots, T_k\}$ *is a pattern on* $\{1, \ldots, n-1\}$ *and that* $\{S_1, \ldots, S_m\}$ *is a pattern on the strategy subset* $\{1, \ldots, r\}$ *where* $r<n-1$ *such that each* T_l *is the same as one of the S's for* $l>j$. *Then the pattern* $\{T_1, \ldots, T_j, S'_1, \ldots, S'_m\}$ *is attainable on* $\{1, \ldots, n\}$, *where* $S'_l = S_l \cup \{n\}$.

Proof. The construction for the proof of this theorem is as follows:

> a'_{ln} and a'_{nl} are defined as in the corresponding sections of Theorem 2 if $l \leqq r$, and $a'_{ln} = -M\gamma$, $a'_{nl} = -\alpha^{n-1}M$ if $l>r$

where α is defined as in previous theorems, $\gamma = 1/(\min p_{jn})$, and the p_{jn}'s are the equilibrium values which are non-zero for the nth strategy on the new matrix $\boldsymbol{A}'$. This theorem is proved in Sect. 6.4.

For example, (using the same initial pattern as for Theorem 2) the pattern $\{(123)(24)(34)\}$ is attained by the matrix

$$\boldsymbol{M18} = \begin{bmatrix} 0 & -1 & 2 & -3 \\ 2 & 0 & -1 & 3 \\ -1 & 2 & 0 & 2 \\ -8 & 7 & 1 & 0 \end{bmatrix}.$$

The subpattern attained by the submatrix corresponding to the strategies 1, 4 is $\{(1)(4)\}$. So that, by Theorem 5, the pattern $\{(123)(24)(34)(15)(45)\}$ is attainable and is attained by the matrix

$$\boldsymbol{M}\,\mathbf{19}=\begin{bmatrix} 0 & -1 & 2 & -3 & 1 \\ 2 & 0 & -1 & 3 & -56 \\ -1 & 2 & 0 & 2 & -56 \\ -8 & 7 & 1 & 0 & 1 \\ 7 & -70/3 & -70/3 & 7 & 0 \end{bmatrix}.$$

4.5.1 Corollary *Theorem 5 generates every pattern that is attainable by cliques matrices.*

Proof. Suppose that Theorem 5 generates every clique pattern up to k strategies. Any clique pattern can be generated by a matrix with

$$a_{ii}=0, \qquad a_{ij}=a_{ji}=\pm 1, \quad j\neq i\,.$$

All ESS's are such that for every pair i, j in the support of the ESS $a_{ij}=a_{ji}=1$. An ESS can only be invaded by row l if $a_{li}=1$ for every i in the support of the ESS.

Now consider a clique pattern on $k+1$ strategies. Let this pattern be $\{S_1,\ldots,S_k,I'_1,\ldots,I'_t\}$ where none of the S_l's contain $k+1$, but all of the I'_l's do, and where $I_l=I'_l\setminus\{k+1\}$. Thus if i is an element of I_l then $a_{k+1\,i}=1$. So that each of the I_l's is a subset of J, where $i\in J$ iff $a_{k+1\,i}=1$. Similarly $a_{k+1\,i}=-1$ for at least one of $i\in S_l$, i.e. none of the S_l's is a subset of J.

Now if we apply Theorem 5 we can add $k+1$ to every support of an ESS on J, and leave every ESS which has support which is not a subset of J the same. If the clique pattern on $\{1,\ldots,k+1\}$ is as above, then the pattern on J is $\{I_1,\ldots,I_t\}$ and the supports of ESS's on $\{1,\ldots,k\}$ which are not subsets of J are the S_l's. This means that Theorem 5 yields the required clique pattern on $\{1,\ldots,k+1\}$. So that the corollary is proved.

Note that Theorem 5 does not **only** generate clique patterns.

The following theorem uses a combination of the methods used in Theorems 4 and 5, which now become special cases of

4.6

Theorem 6 *Suppose $\{T_1,\ldots,T_r\}$ is a pattern on a payoff matrix on $\{1,\ldots,n-1\}$ and that $\{S_1,\ldots,S_j\}$ is a pattern on $\{1,\ldots,k\}$ such that each of the S_l's is a subset of $\{1,\ldots,i\}$ where $i\leqq k\leqq n-1$. T_l is a subset of $\{1,\ldots,k\}$ for $l>s$ but not for $l\leqq s$, $s\leqq r$ and $i\leqq k\leqq n-1$. Suppose further that I is a subset of $\{i+1,\ldots,k\}$, that T_l is not a subset of I for $l\in\{1,\ldots,s\}$ and that the submatrix $\boldsymbol{A}_I$ of $\boldsymbol{A}$ corresponding to the set I satisfies the negative definiteness condition. Then the pattern $\{T_1,\ldots,T_s, S'_1,\ldots,S'_j,I'\}$ is attainable on $\{1,\ldots,n\}$ where $I'=I\cup\{n\}$, and $S'_l=S_l\cup\{n\}$.*

The construction of the new matrix $\boldsymbol{A}'$ together with the proof of the theorem is in Sect. 6.5.

Example. The pattern $\{(123)(234)\}$ is attained by the matrix

$$\boldsymbol{M}\mathbf{20}=\begin{bmatrix} 0 & -1 & 9 & 0.5 \\ 2 & 0 & 1 & 9 \\ -4 & 2 & 0 & 9 \\ 2 & -2 & 10 & 0 \end{bmatrix}.$$

The pattern attained by the submatrix associated with 1, 2, 4 is $\{(2)\}$ and the submatrix corresponding to 1, 4 is negative definite. Then by Theorem 6 the pattern $\{(123)(234)(145)(25)\}$ exits on $\{1,\ldots,5\}$. Using the construction in Sect. 6.5, it is attained by the matrix

$$\boldsymbol{M}\mathbf{21}=\begin{bmatrix} 0 & -1 & 9 & 0.5 & 1 \\ 2 & 0 & 1 & 9 & 0.95 \\ -4 & 2 & 0 & 9 & -4000 \\ 2 & -2 & 10 & 0 & 0.9985 \\ 500.24925 & 10 & -4000 & 500.25075 & 0 \end{bmatrix}.$$

5 Pattern generation for up to five strategies

The diagram shows how patterns on higher numbers of strategies are generated from patterns on lower numbers of strategies, when the number of strategies equals one up to five. The only patterns considered are maximal and non-degenerate. All degenerate patterns and all known non-maximal patterns on one to five strategies can be attained by using these theorems, but the maximal non-degenerate patterns are the more complex and difficult to find in general, and so are the more interesting. Where a pattern can be found by using Theorem 1 or Theorem 2a) it is stated (these are the simplest of the theorems). Another method will only be given for a pattern which cannot be attained by either of these two theorems.

All known maximal patterns on five strategies can be obtained by applying the methods in this paper to the patterns on four strategies, except $\{(123)(234)(345)(451)(512)\}$ (all non-maximal patterns can also be found in this way). All patterns on four strategies can be found using these methods on the patterns on three strategies, and similarly for the patterns on three strategies and two strategies. However this does not mean that all these patterns can be generated starting from one strategy using these methods; the construction for $\{(12)(13)\}$ given by using Theorem 1 on (12) will not yield $\{(12)(13)(234)\}$ under Theorem 4. Theorems 4–6 require additional conditions, as well as the pattern on the original payoff matrix, which are not satisfied by this construction.

The problem with the pattern $\{(123)(234)(345)(451)(512)\}$ (as with some of the patterns as yet unresolved on five strategies) is that each pair of strategies occurs in the support of at least one ESS. The only Adding Theorem which can generate such a pattern is Theorem 2a), which adds the new element to all existing ESS's. So any pattern with every strategy pair in some ESS but no one element in every ESS cannot be attained by use of one of these theorems. Patterns of this type become more common as the number of strategies increases, so that the number of 'holes' in the diagram will increase as the number of strategies does.

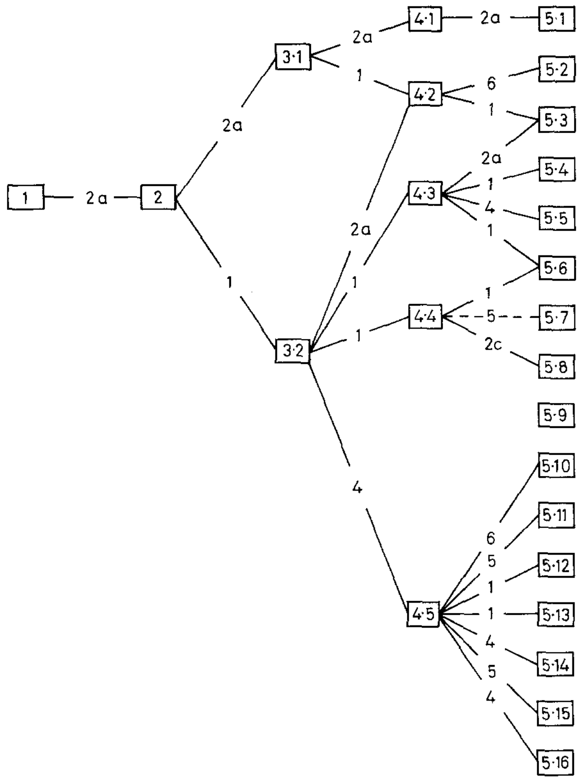

Fig. 1

The numbers in the diagram represent patterns as follows:

1 – {(1)}
2 – {(12)}
3.1 – {(123)}
3.2 – {(12)(13)}
4.1 – {(1234)}
4.2 – {(123)(124)}
4.3 – {(12)(13)(14)}
4.4 – {(12)(23)(34)(14)}
4.5 – {(123)(14)(24)}
5.1 – {(12345)}
5.2 – {(123)(234)(145)(25)}
5.3 – {(123)(124)(125)}
5.4 – {(12)(13)(14)(15)}
5.5 – {(1234)(15)(25)(35)}
5.6 – {(14)(15)(24)(25)(34)(35)}
5.7 – {(12)(23)(34)(45)(51)}
5.8 – {(123)(234)(345)(15)}
5.9 – {(123)(234)(345)(451)(512)}

5.10 – {(123)(14)(24)(15)(35)}
5.11 – {(123)(14)(24)(35)(45)}
5.12 – {(123)(14)(24)(15)(25)}
5.13 – {(123)(234)(15)(25)(45)}
5.14 – {(123)(234)(25)(35)}
5.15 – {(123)(345)(14)(25)}
5.16 – {(123)(345)(14)(15)}

The dotted line on the diagram indicates a split from a subpattern of 4.4, {(12)(23)(34)}. Pattern 5.9 cannot be found from any of the 4-strategy patterns using any of the methods.

Conclusion

The methods described in this paper provide new ways of finding new patterns of ESS's using known patterns on a small number of strategies. It is thus possible to build up complex patterns quite easily. However, it is not possible to find all patterns using these methods, and as the number of strategies increase, more and more of the interesting/complicated patterns will not be generated using these methods. Indeed for even small n the attainability of some patterns is unknown (the patterns {(1234)(125)(345)} and {(1234)(125)(235)(345)} on five strategies have only recently been shown to be attainable, the attainability of other patterns on five strategies is still unknown). Many interesting problems remain and are proving surprisingly intractible, whilst those that are solved often throw up new questions.

6 Proofs

6.1 Proof of Theorem 2

To prove this theorem we need an additional result, namely that if $\boldsymbol{A}$ is a $(k-1)\times(k-1)$ matrix which satisfies the negative definiteness condition, then $\boldsymbol{B}=(b_{ij})$, which is $k\times k$, satisfies the same condition, where $\boldsymbol{B}$ is defined as follows:

X1: $b_{ij}=a_{ij} \quad \forall i,j\in\{1,\ldots,k-1\}$
$b_{ik}=\alpha_i,\ b_{ki}=r-\alpha_i \quad \forall i\in\{1,\ldots,k-1\}$ and r sufficiently large: $b_{kk}=0$
X2: $b_{ij}=a_{ij} \quad \forall i,j\in\{1,\ldots,k-1\}$
$b_{ik}=r,\ b_{ki}=\alpha_i \quad \forall i\in\{1,\ldots,k-1\}$: $\quad b_{kk}=0$, for sufficiently large r.

Note that the α_i are arbitrary.

Proof of X1. Let $\boldsymbol{y}$ be any k-vector whose elements sum to zero. Set $\boldsymbol{y}=(z,\alpha)$ where $\sum z_i=-\alpha$. Let $\boldsymbol{A}^1=(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})/2$ and define $\boldsymbol{s}$ by $(\boldsymbol{A}^1\boldsymbol{s})_i=1$ for $\forall i\in\{1,\ldots k-1\}$. Put $x_i=(\sum s_j)z_i+\alpha s_i \ \forall i\in\{1,\ldots,k-1\}$, then $\sum x_i=0$. Let $S=\sum s_j$ then $0>\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^1\boldsymbol{x}=\sum x_i(S(\boldsymbol{A}^1\boldsymbol{z})_i+\alpha(\boldsymbol{A}^1\boldsymbol{s})_i)=\sum x_i(S(\boldsymbol{A}^1\boldsymbol{z})_i+\alpha)=S\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^1\boldsymbol{z}=S(S\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}^1\boldsymbol{z}+\alpha\boldsymbol{s}^{\mathrm{T}}\boldsymbol{A}^1\boldsymbol{z})=S(S\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}+\alpha\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}^1\boldsymbol{s})=S(S\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}+\alpha\sum z_i)$. Thus $0>S(S\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}-(\sum z_i)^2)$ and hence $\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}<(\sum z_i)^2/\sum s_i$.

Now $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{y}=\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}-r(\sum z_i)^2<(\sum z_i)^2(1-rS)/S$, so that the new matrix satisfies the negative definiteness condition if $r>1/S$.

The proof fails if Det $(A^1)=0$, for then such an s may not exist. In this case there is a solution to $(A^1 s)_i=0\ \forall i\in\{1,\ldots,k-1\}$ so that $0>x^T A x=S^2 z^T A z$.

Now $y^T B y=z^T A z-r(\sum z_i)^2$ so if $r>0$, then $y^T B y<0$, i.e. B satisfies the negative definiteness condition.

Proof of X2. Let q be a k-vector whose elements sum to zero. Let w be a k-vector which comprises the first $k-1$ elements of q, together with a zero in the kth position. Then $q^T B q=w^T B w+q_k(\sum w_i)r+q_k\sum w_i\alpha_i=w^T B w-q_k^2 r+q_k(\sum w_i\alpha_i)$. Let x be a k-vector, which kth element zero. There exists a positive h such that whenever $\max(|x_i|)=1$ and $\sum x_i=0$, $x^T B x<-h$. (If this were not true we could find a sequence of x's such that the limit of $x^T B x$ is zero, and since the limit of the x's has elements which sum to zero although it is not identically zero, B would not satisfy the negative definiteness condition).

Suppose w is such that $\max(|x_i|)=1$ and set $x_i=w_i\ \forall i\in\{1,2,\ldots,k-2\}$ and $x_{k-1}=w_{k-1}+q_k$, then $\sum x_i=0\Rightarrow\sum w_i=-q_k$, $w^T B w<-h+|q_k|(\sum_{i<k-1}|a_{k-1i}+a_{ik-1}|)$. Considering such w's is sufficient, since any $k-1$ vector can be expressed as a constant multiplied by a vector whose maximum element is 1, and the constant cannot affect the sign of $w^T B w$. Thus

$$\begin{aligned} q^T B q &\leqq -h+|q_k|(\textstyle\sum_{i<k-1}|a_{k-1i}+a_{ik-1}|)-q_k^2 r+q_k\sum w_i\alpha_i \\ &< -h-(q_k)^2 r+\gamma|q_k|, \end{aligned}$$

where $\gamma=\sum|\alpha_i|+\sum_{i<k-1}|a_{k-1i}+a_{ik-1}|$. Now γ and h are constant, irrespective of the vector q. Let r be greater than $\gamma^2/(4h)$. Then $q^T B q$ is always negative for any q whose elements sum to zero (but $q\neq 0$). Hence B satisfies the negative definiteness condition.

Proof of theorem. a)

(i) $M=\sup(a_{ij})$ so that row n dominates all other rows (excluding column n), so n invades every ESS on A, so all ESS's on A' involve n.

(ii) Suppose $p=(p_1,\ldots,p_i)$ is an equilibrium on a submatrix of A, C i.e. $(Cp)_j=\lambda\ \forall j\in\{1,\ldots,i\}$ (choosing the support of p to be $\{1,\ldots,i\}$ without loss of generality).

Let $r=(tp_1,\ldots,tp_i,(1-t))$ so that $\sum r_s=1$, and let $t=1/(M-\lambda+1)$. Let C' be the payoff matrix associated with the set of strategies $\{1,\ldots,i,n\}$ then $(C'r)_j=t(Cp)_j+(1-t)=t\lambda+(1-t)=M/(M-\lambda+1)\quad\forall j\in\{1,\ldots,i\}$ and $(C'r)_n=Mt=M/(M-\lambda+1)$.

Now $M>\lambda$ since n invades all ESS's, so $0<t<1$ and all elements of r are positive which implies that r is an equilibrium on $\{1,\ldots,n\}$.

(iii) It is easy to show that strategy k invades r if and only if it invades p. $E[k,r]=t(Cp)_k+(1-t)<E[r,r]$ iff $(Cp)_k<\lambda$ i.e. k invades r if and only if it invades p, so the equilibrium on $\{1,\ldots,i,n\}$ is invaded on $\{1,\ldots,n-1\}$ iff the equilibrium on $\{1,\ldots,i,n\}$ is invaded on $\{1,\ldots,n\}$.

(iv) By result X1, if C is negative definite, then so is C'.

From (i)–(iv) we can see that if S_j is the support of an ESS on $\{1,\ldots,n-1\}$ then S_j', which is the same as $S_j\cup\{n\}$, is the support of an ESS on $\{1,\ldots,n\}$. Similarly using a reverse argument on (ii) and (iii) (together with the fact that every submatrix of a matrix which satisfies the negative definiteness condition satisfies the condition itself) we can see that if any other ESS exists on $\{1,\ldots,n\}$

containing n then there exists a corresponding ESS on $\{1, \ldots, n-1\}$ with n removed. Hence Theorem 2a) is proved.

b) Suppose S_j contains none of $\{1, \ldots, i\}$, then following the same argument as in a), we can see that S'_j is the support of an ESS on $\{1, \ldots, n\}$ iff S_j is the support of an ESS on $\{1, \ldots, n-1\}$.

If S_j contains one or more of $\{1, \ldots, i\}$ and is the support of an ESS then n cannot invade, because one or more of the entries a_{nl} in the row trying to invade is $-\alpha M$ which dominate(s) the other entries. So S_j is still the support of an ESS. If S_j is not the support of an ESS on $\{1, \ldots, n-1\}$ then $S_j \cup \{n\}$ is not the support of an ESS on $\{1, \ldots, n\}$ for the same reasons as in a). The pure $\{n\}$ is not the support of an ESS on $\{1, \ldots, n\}$. So Theorem 2b) is proved.

c) Suppose that S_j is disjoint from $\{1, \ldots, i\}$ and is the support of an ESS on $\{1, \ldots, n-1\}$. Then $E[n, \boldsymbol{p}] = 0 < E[\boldsymbol{p}, \boldsymbol{p}]$ where $\boldsymbol{p}$ is the ESS on S_j, so that n does not invade $\boldsymbol{p}$ i.e. S_j is the support of an ESS on $\{1, \ldots, n\}$. Using the same argument as in a) and b) if S_j is not the support of an ESS on $\{1, \ldots, n-1\}$ then $S_j \cup \{n\}$ cannot be the support on an ESS on $\{1, \ldots, n\}$. Now suppose that S_j does contain at least one element of $\{1, \ldots, i\}$. At least one of the elements in the invading row is αM which means that $E[n, \boldsymbol{p}]$ is at least M, so that n invades $\boldsymbol{p}$.

Using a similar argument to a)(ii) we can show that S'_j has an equilibrium on it of the form $(t\boldsymbol{p}, (1-t))$ for some $t \in (0, 1)$ since n invades $\boldsymbol{p}$ iff $\boldsymbol{p}$ is an equilibrium. In the same way $\boldsymbol{r}$, the new equilibrium, can be invaded by n iff $\boldsymbol{p}$ can be. Finally, by X2, the corresponding submatrix to S'_j satisfies the negative definiteness condition iff the submatrix corresponding to S_j does. Hence Theorem 2c) is proved.

d) Suppose S_j is the support of an ESS, $\boldsymbol{p}$ on $\{1, \ldots, n-1\}$ so that $E[n, \boldsymbol{p}] = \sum_{(l \in S_j)} a_{nl} p_l$. Each $l \in S_j$ is a member of some U_s. Let k be given by $k = \max_{l \in S_j}(s\colon l \in U_s)$. The values of a_{nl} have been constructed in such a way that the invasion row is dominated by the values of a_{nl} for which $l \in U_k$. So that if k is odd n invades, and if k is even n does not invade.

By X2 the new submatrix corresponding to S'_j satisfies the negative definiteness condition, and using the same arguments as before existence of an equilibrium and invadability by other rows is unchanged. In the same way no other ESS can exist on A', so that Theorem 2d) is proved.

6.2 *Proof of Theorem 3*

$\boldsymbol{A} = (a_{ij})$ is the pay-off matrix defined on strategies $\{1, \ldots, n-2\}$. The construction used to prove Theorem 3 is as follows:

$$a'_{ij} = a_{ij} \quad \forall i, j \in \{1, \ldots, n-2\}$$

$$a'_{n-1\,j} = a_{1j}, a'_{jn-1} = a_{j1} \quad \forall j \in \{2, \ldots, n-2\}$$

$$a'_{nj} = a_{2j}, a'_{jn} = a_{j2} \quad \forall j \in \{1, 3, 4, \ldots, n-2\}$$

$$a'_{n-1\,1} = a'_{1\,n-1} = a'_{n2} = a'_{2n} = -(2\alpha)^2 M$$

$$a'_{n-1\,n-1} = a'_{nn} = 0, a'_{n-1\,n} = a'_{nn-1} = 2\alpha M$$

where $M = \max |a_{st}|$ $s, t = 1, \ldots, n-2$, $\alpha = 1/\min(p_{st})$, and p_{st} is the value of the tth non-zero element of the sth ESS of $\boldsymbol{A}$.

Now if we exclude columns n and $n-1$, row 1 dominates row $n-1$ and row 2 dominates row n and so all ESS's of A are still ESS's of A'. Let A^+ be the submatrix of A' associated with elements $\{2, \ldots, n-1\}$. Then A^+ is just the same matrix as A with 1 relabelled $n-1$, so for every ESS on A there is an ESS of A^+ with 1 replaced by $n-1$. Now if we exclude columns 1 and n, we can see that row $n-1$ dominates row 1, so that row 1 cannot invade any ESS of A^+. We now show that n invades any ESS of A^+ if and only if its support contains 2. Recall that $a'_{nn-1} = 2\alpha M$, $a'_{n2} = -(2\alpha)^2 M$ and so if $\boldsymbol{p}$ is an ESS with support S_j which does not contain 2 then $E[n, \boldsymbol{p}] > 2M - M = M$ so n invades $(E[\boldsymbol{p}, \boldsymbol{p}] < M)$.

If S_j does contain 2, then $E[n, \boldsymbol{p}] < 2\alpha M - (2\alpha)^2 M + M < -\alpha M$ so strategy n cannot invade. So $\boldsymbol{p}$ is an ESS iff $2 \in S_j$.

This means that if A has an ESS whose support S_j contains both 1 and 2, another ESS exists on A' with 1 replaced by $n-1$. Using the same argument we find that if S_j of A contains both 1 and 2, another ESS exists on A' with 2 replaced by n. $\{n-1, n\}$ is the support of an ESS $(E[\boldsymbol{p}, \boldsymbol{p}] = 2\alpha M > 2M > E[i, \boldsymbol{p}] \forall i \in \{1, \ldots, n-2\})$. No ESS can have a support which contains both 1 and $n-1$ or both 2 and n, or both $n-1$ and n, except $\{n-1, n\}$ itself, since it is the support of an ESS. Using a similar argument to that in Theorem 1, no other ESS's can exist on A', so Theorem 3 is proved.

6.3 *Proof of Theorem 4*

Let the matrix A' be defined as follows:

$$a'_{ij} = a_{ij} \quad \forall i, j \in \{1, \ldots, n-1\}$$

$$a'_{in} = b_i, a'_{ni} = h - b_i \quad \forall i \in I$$

$$a'_{in} = a'_{ni} = -M, \quad \forall i \in I, \qquad a'_{nn} = 0\ ,$$

where M is sufficiently large and h and the b_i arbitrary .

Define r_i as $\sum_{(j \in I)} a_{ij}$, let $|I|$ be the size of I and let $b_i = b_1 + (\alpha/(1-|I|\alpha))(r_1 - r_i)$ $\forall i \in I$. Consider $\boldsymbol{p} = (\alpha, \alpha, \ldots, \alpha, \ 1-|I|\alpha)$ a vector on I'. Then $E[i, \boldsymbol{p}] = \sum_{(j \in I)} (\alpha a_{ij}) + b_i(1-|I|\alpha) = \alpha r_i + (r_1 - r_i)\alpha + b_1(1-|I|\alpha) = \alpha r_1 + b_1(1-|I|\alpha)$ for $i \in I$ and $E[n, \boldsymbol{p}) = \alpha h|I| - \alpha \sum_{i \in I} b_i$ which can be shown to equal $\alpha r_1 + b_1(1-|I|\alpha)$ when $h = 1/(\alpha|I|)[b_1 + \alpha r_1 + \alpha^2/(1-\alpha|I|) \sum_{i \in I}(r_1 - r_i)]$. So for $0 < \alpha < 1/|I|$, $\boldsymbol{p}$ is an equilibrium on I'.

For large enough M, $\boldsymbol{p}$ cannot be invaded by any i not in I. By making b_1 large or α small we can make h sufficiently large to ensure that, by X1 in Sect. 6. B, if A_I satisfies the negative definiteness condition so does $A_{I'}$, and hence that $\boldsymbol{p}$ is an ESS of $\{1, \ldots, n\}$ with support I'.

If S_j is the support of an ESS of $\{1, \ldots, n-1\}$, then there is at least one $l \in S_j$ which is not an element of I. Thus $a_{ni} = -M$, where M is large, so that strategy n cannot invade the ESS on S_j i.e. S_j is the support of an ESS on $\{1, \ldots, n\}$.

No ESS's can contain both an i not belonging to I and $\{n\}$, because the corresponding submatrix would not satisfy the negative definiteness condition. Thus all ESS's exist either on $\{1, \ldots, n-1\}$ or on I'. But I' is itself the support of an ESS, so by Bishop-Cannings there can be no other ESS's on I', and since we already know that the pattern on $\{1, \ldots, n-1\}$ is $\{S_1, \ldots, S_k\}$ no other ESS's exist. The proof of Theorem 4 is now complete.

6.4 *Proof of Theorem 5*

Using the same reasoning as in Theorem 3, (since the submatrix of the payoff matrix associated with $\{1, \ldots, i, n\}$ is the same as that used in Theorem 3), if S_j is the support of an ESS on $\{1, \ldots, i\}$ then S'_j is the support of an ESS on $\{1, \ldots, i, n\}$. Suppose that the ESS associated with S_j is $\boldsymbol{p}$, then if S'_j contains n (if $S'_j = S_j$ then trivially k cannot invade)

$$E[k, \boldsymbol{p}] = \textstyle\sum_{i \in S'_j} a_{ki} p_i = \sum_{i \in S_j} a_{ki} p_i - p_n \beta M < M - M = 0 \quad \text{for } k \notin \{1, \ldots, i, n\}$$

i.e. S'_j is the support of an ESS on $\{1, \ldots, n\}$. Each T_j is the support of an ESS on $\{1, \ldots, n-1\}$. As before the a_{nl}'s have been chosen to be sufficiently small for l not a member of $\{1, \ldots, i\}$ such that any support which contains such an l cannot be invaded (this is easy to check), so that T_j is an ESS on $\{1, \ldots, n\}$ unless it is equal to one of the S_j's.

No support of an ESS can contain both n and j for $i < j < n$ so that all ESS's occur on either $\{1, \ldots, n-1\}$ or $\{1, \ldots, i, n\}$. But we know the pattern for $\{1, \ldots, n-1\}$ and the submatrix on $\{1, \ldots, i, n\}$ is the same as that in Theorem 3, so that no other ESS's exist on $\{1, \ldots, i, n\}$ either. Hence there are no other ESS's of $\{1, \ldots, n\}$. So Theorem 5 is proved.

6.5 *Proof of Theorem 6*

We have a matrix A defined on $\{1, \ldots, n-1\}$. Define the matrix A' on $\{1, \ldots, n\}$ as follows.

$$a'_{ij} = a_{ij} \quad \forall i, j \in \{1, \ldots, n-1\}$$

$$a'_{jn} = V,\ a'_{nj} = L \quad \forall j \in \{1, \ldots, i\}$$

$$a'_{jn} = b_j,\ a'_{nj} = K - b_j \quad \forall j \in \{i+1, \ldots, k\}$$

$$a'_{jn} = a'_{nj} = -M \quad \forall j \in \{k+1, \ldots, n-1\}$$

$$a'_{nn} = 0$$

where b_j is defined as in the proof of Theorem 4.

Consider ESS's on $\{1, \ldots, k\}$ which only include elements of $\{1, \ldots, i\}$ in their supports. Let $\boldsymbol{p}$ be such an ESS. Then $\boldsymbol{p}' = (\gamma \boldsymbol{p}, 1-\gamma)$ is such an ESS on $\{1, \ldots, i, n\}$ from previous proofs (where $\gamma = V/(V+L-\lambda)$, $\lambda = E[\boldsymbol{p}, \boldsymbol{p}]$). If M is very large compared to every other element of A', then none of $\{k+1, \ldots, n-1\}$ can invade. We show that under certain conditions s, $i < s \leq k$, cannot invade $\boldsymbol{p}'$.

Now $E[\boldsymbol{p}', \boldsymbol{p}'] = LV/(V+L-\lambda)$ and $E[s, \boldsymbol{p}'] = \gamma E[s, \boldsymbol{p}] + (1-\gamma) b_s = \gamma(\lambda - \delta_s) + b_s(1-\gamma)$, (where δ_s is positive, since $\boldsymbol{p}$ is an ESS on $\{1, \ldots, k\}$) $= (V\lambda - V\delta_s + b_s(L-\lambda))/(V+L-\lambda)$.

So to stop s invading, we need $VL > V\lambda - V\delta_s + b_s(L-\lambda)$. If $V < b_s$ then the right-hand side of this inequality is less than $b_s L - V\delta_s$, so if $VL > b_s L - V\delta_s$ then s cannot invade i.e. if $\delta_s > (b_s - V)L/V$.

So $\boldsymbol{p}'$ is certainly an ESS on $\{1, \ldots, n\}$ if $\delta > (b-V)L/V$, where δ is the smallest of the δ_s's and b is the largest of the b_s's.

Strategy n cannot invade any ESS whose support contains $j \in \{k+1, \ldots, n-1\}$, because of the $-M$'s, so the S_j's are still supports of ESS's.

Now because of the definition of the b_j's $\boldsymbol{p}=(t, t, \ldots, t, 1-(k-i)t)$ is an ESS with support $\{i+1, \ldots, k, n\}$. Then $E[\boldsymbol{p}, \boldsymbol{p}]=t(k-i)K-\sum_{j=i+1}^{k} b_j$ where

$$K=\frac{1}{t(k-i)}\left(b_{i+1}+tr_{i+1}+\frac{t}{1-(k-i)t}\sum_{j=i+1}^{k}(r_{i+1}-r_j)\right).$$

This expression tends to b_{i+1} as t tends to zero (as t tends to zero, all the b_j's converge to b_{i+1}).

Also $E[s, \boldsymbol{p}]=(\sum_{j=i+1}^{k} a_{sj})t+(1-(k-i)t)V$ $\forall s\in\{1, \ldots, i\}$ which converges to V as t tends to zero. But V is smaller than b_{i+1}, so that for some sufficiently small t, $\boldsymbol{p}$ cannot be invaded by any of $1, \ldots, i$. None of the strategies $k+1, \ldots, n-1$ can invade because of the $-M$'s in the nth column (the strategy for which the entry in $\boldsymbol{p}$ is tending to unity), so that $1, \ldots, i, n$ is the support of an ESS on A'.

Let $b_{i+1}=V+1$ then for small enough t, b (the maximum of the b_s's) is less than $V+2$. So that $(a-V)L/V<2L/V=2/V^{1/2}$ if $L=V^{1/2}$. Now there are a finite number of ESS's on $\boldsymbol{A}$ each with their own value of $\delta>0$. So if V is large enough $2/V^{1/2}<\delta$ for each such δ, i.e. if S_l is the support of an ESS on $\{1, \ldots, k\}$ which includes only strategies in $\{1, \ldots, i\}$, then $S_l'=S_l\cup\{n\}$, is the support of an ESS on $\{1, \ldots, n\}$.

In the above argument use is made of certain values being large compared to others; all these arguments are indeed consistent. We have L large compared to a_{ij} for $i, j\in\{1, \ldots, n-1\}$, $V=L^2$, the b_s's are slightly larger than V. K is large compared to V. Finally M is large compared with everything else. Thoerem 6 is proved.

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