Recall that for any positive integer n we denote by S_n the symmetric group of degree n, by A_n the alternating group of degree n, and by $\mathbb{Z}_n$ the group $\mathbb{Z}_n = \{0, 1, , \ldots, n-1\}$ together with addition modulo n.

1. (a) **(4 marks)** Give the definition of a group.

   (b) **(2 marks)** Explain why the set $A = \{1, 2, 3, 4, 5\}$ together with multiplication modulo 6 is not a group.

   (c) **(4 marks)** Prove that the set $B = \{1, 2, 3, 4\}$ together with multiplication modulo 5 is a group.

   (d) **(1 marks)** List all the elements of the group C consisting of all rotational symmetries of a square.

   (e) **(4 marks)** Prove that the groups B and C given above are isomorphic.

   (f) **(3 marks)** Explain why B and C cannot be isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

   (g) **(2 marks)** Find a homomorphism $\theta : B \to C$ which is not an isomorphism.

2. Let $(G, *)$ be a finite group with identity element e.

   (a) **(2 marks)** Explain what is meant by the order of an element $g \in G$.

   (b) **(2 marks)** State a result linking the order of any $g \in G$ to the order of the group G.

   (c) **(4 marks)** Suppose that $|G| = 6$. Let $g \in G$ with $g^7 = e$. What can you deduce about g?

   (d) **(12 marks)** Suppose that $G = \{e, u, v, w, x, y\}$ where w is an element of order 2 and $u*v = x$. Suppose further that we have an isomorphism

$$\phi : S_3 \to G$$

   satisfying $\phi(1, 2) = u$ and $\phi(1, 3) = v$. Determine $\phi(g)$ for all $g \in G$. Hence, write down the Cayley table for the group G.

Turn over ...

3. (a) **(3 marks)** When do we say that a subset H of a group G is a subgroup of G?

   (b) **(7 marks)**

      i. State Lagrange's theorem for a finite group G.

      ii. List all subgroups of $\mathbb{Z}_{17}$.

      iii. List all subgroups of $\mathbb{Z}_9$.

   (c) **(10 marks)**

      i. Let H be a subgroup of $\mathbb{Z}_9$ of order 3. Explain why it is a normal subgroup of $\mathbb{Z}_9$.

      ii. Find the cosets of H in $\mathbb{Z}_9$ and write down the Cayley table for $\mathbb{Z}_9/H$.

      iii. Consider the homomorphism

$$\phi : \mathbb{Z}_9 \longrightarrow \mathbb{Z}_3 : n \mapsto n \text{ modulo } 3$$

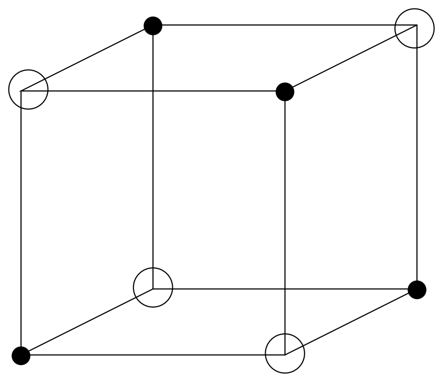
mapping $n \in \mathbb{Z}_9$ to its residue modulo 3. Find the kernel and the image of ϕ. Deduce that $\mathbb{Z}_9/H \cong \mathbb{Z}_3$.

4. (a) **(8 marks)** Describe the group G consisting of all rotational symmetries of a cube. What is the order of G?

   (b) **(12 marks)** How many different cubes can be constructed by painting each face of a cube red, white or blue?

Turn over ...

5. The vertices of a cube are coloured black or white as in the picture below. Let G be the group of all rotational symmetries of this coloured cube.



(a) **(2 marks)** State the Orbit-Stabilizer theorem.

(b) **(4 marks)** Use (a) to show that the order of G is equal to 12.

(c) **(3 marks)** State the classification of all 3-dimensional finite rotation groups.

(d) **(3 marks)** Use (b) and (c) to deduce that G is isomorphic to A_4.

(e) **(8 marks)** By considering the action of G on the black vertices of the cube, construct an explicit isomorphism from G to A_4.

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