Recall that for any positive integer n we denote by S_n the symmetric group of degree n, by A_n the alternating group of degree n, and by $\mathbb{Z}_n$ the group $\mathbb{Z}_n = \{0, 1, , \ldots, n-1\}$ together with addition modulo n.

1. Let $G = \{e, x, y, z, u, v\}$ be a group with multiplication $*$ and identity element e. Suppose that G is abelian. Suppose further that $x^2 = y^2 = z$, $z^2 = v^2 = x$, $x * y = v$, $x * z = e$, $u * x = y$ and u has order 2.

   (a) **(6 marks)** Write down the Cayley table for G.

   (b) **(3 marks)** Define what is meant by the order of a group and by the order of an element of a group. State a result which relates these two orders.

   (c) **(2 marks)** Find the inverse and the order of each element in G.

   (d) **(9 marks)** Can G be isomorphic to any of the following groups? Justify your answers in each case by finding an explicit isomorphism or explaining why no such isomorphism exists.

   i. S_6 ii. S_3 iii. $\mathbb{Z}_6$ iv. $\mathbb{Z}_2 \times \mathbb{Z}_3$

2. (a) **(2 marks)** When do we say that a subset H of a group G is a subgroup of G?

   (b) **(2 marks)** When do we say that a subgroup H of a group G is a normal subgroup?

   (c) **(8 marks)** Find the left cosets and the right cosets of H in G in each of the following cases.

   i. $G = S_3$ and $H = \langle (1, 2) \rangle$.

   ii. $G = S_3$ and $H = \langle (1, 2, 3) \rangle$.

   iii. $G = \mathbb{Z}$ and $H = \langle 10, 15 \rangle$.

   Among the three subgroups H given above, precisely two of them are normal subgroups of G. Which ones are they?

   (d) **(8 marks)** For each of the two normal subgroups H given in (c), write down the Cayley table for G/H and find a 'known' group G' such that $G/H \cong G'$.

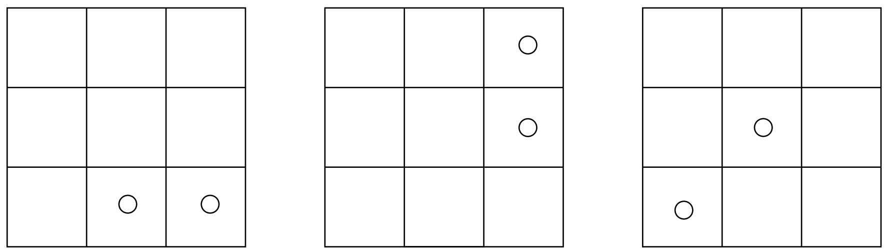
Turn over ...

3. (a) **(2 marks)** Let G be a finite group acting on a finite set X. Define what is meant by Fix(g) for some $g \in G$.

   (b) **(2 marks)** State Burnside's Counting theorem.

   (c) i. **(12 marks)** Suppose that it is proposed to manufacture ID cards from plastic squares, marked with a 3×3 grid on both sides and punched with two holes (see examples in the pictures below). How many different ID cards can be produced in this way?

   ii. **(4 marks)** Draw all the different ID cards.



4. (a) **(2 marks)** State the Orbit-Stabilizer Theorem.

(b) **(2 marks)** Let G be the group of all rotational symmetries of a cube. Using (a), find the order of G.

(c) **(6 marks)** Suppose that the faces of the cube are now painted in some way (with colours or drawings) and let G' be the group of all rotational symmetries of the painted cube.

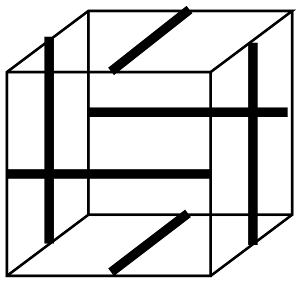
What is the relationship between G and G'?

Is it possible to paint the cube in such way as to have

i. $|G'| = 9$? ii. $G' = \{e\}$?

If it is, give an example and if it is not, explain why not.

(d) Consider the painted cube obtained by drawing a straight line across each face in such a way that no two lines meet (see picture below). Let G' be the rotational symmetry group of this painted cube.



i. **(2 mark)** By considering the action of G' on the set of faces of the cube, show that $|G'| = 12$.

ii. **(2 marks)** State the classification of all finite rotational symmetry groups of 3-dimensional figures.

iii. **(6 marks)** Using i. and ii. show that G' is isomorphic to A_4.

Internal Examiner:	Dr M. De Visscher
External Examiners:	Professor J. Billingham
	Professor E. Corrigan