MA3615 Groups and Symmetry

Exercise Sheet 1: Definition of a Group

1. Describe all the symmetries (rotations and reflections) of a square and write down the Cayley table for this symmetry group G. What is the order of G? Is G abelian? Find the inverse of each element in G.

2. Which of the following are groups? For those which are not group, explain why not (it is enough to show that one axiom fails); for those which are, show that all the axioms are satisfied.

   (a) $(\mathbb{Z}, \times)$

   (b) the set $2\mathbb{Z}$ of all even integers with addition

   (c) the set of all odd integers with addition

   (d) $(\mathbb{Q}, \times)$

   (e) The set of all strictly positive rational numbers with multiplication.

   (f) $\{1, 2, 3, 4\}$ with multiplication modulo 5

   (g) $\{1, 2, 3, 4, 5\}$ with multiplication modulo 6

   (h) $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ with addition of vectors modulo 2

   (i) $\{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},\ m \in \mathbb{Z}_5\}$ with multiplication of matrices.

3. For each *group* in the previous question, decide whether or not it is abelian, and find its order.

4. Let x and y be any elements of a group G and suppose that e is the identity of G. Prove the following statements.

   (a) $(x * y)^{-1} = y^{-1} * x^{-1}$

   (b) $x * y = e$ implies $y * x = e$

   (c) $(x * y)^2 = x^2 * y^2$ implies $x * y = y * x$

5. Let $G = \{e, a, b, c\}$ be a group with identity e and such that $a^2 = b^2 = c^2 = e$. Write down the Cayley table for G.

6. Explain why the following table cannot be the Cayley table of a group.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	e	d	c
b	b	c	d	a	e
c	c	d	a	e	b
d	d	e	c	b	a