# MA3615 Groups and Symmetry

## Exercise Sheet 2: Permutation groups; Isomorphisms

1. Consider the following permutations in $S_7$.

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 2 & 3 & 7 & 5 & 6 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 4 & 3 & 5 & 1 & 7 \end{bmatrix}$$

   (a) Write $\alpha$ and $\beta$ as a product of disjoint cycles.

   (b) Compute $\alpha \circ \beta$, $\beta \circ \alpha$ and $\alpha^2$ giving your answer in cycle form.

   (c) Decide whether either of $\alpha$ or $\beta$ is an even permutation.

2. Show that an $n$-cycle is an even permutation if and only if $n$ is odd. Hence determine which of the following permutations belong to $A_{11}$.

   (a) $(1,2)(3,6,8)(4,11,10,5,9,7)$

   (b) $(1,3,5,7,9,11,2,4,6,8)$

   (c) $(1,2,3,4)(1,2,4,3)(1,4,2,3)$

3. Show that $V_4 = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ form a group of permutations.

4. Let $H = \{u, v, w, x, y, z\}$ be a group (with multiplication $*$) and $\theta : S_3 \to H$ be an isomorphism with $\theta(e) = u$, $\theta(1,2) = x$, $\theta(1,3) = y$, $\theta(2,3) = z$,$\theta(1,2,3) = v$ , $\theta(1,3,2) = w$. Find $x * w$, $w^{-1}$, $v^5$, $z * v^{-1} * x$.

5. Show that the group $V_4$ described above is isomorphic to the group of symmetries of a (non-square) rectangle. Can you find another group among the examples in Exercise Sheet 1 which is also isomorphic to $V_4$?

6. Classify the following groups into isomorphism classes.

   (a) the group of rotational symmetries of an equilateral triangle

   (b) the group of all symmetries of an equilateral triangle

   (c) $S_3$

   (d) $\mathbb{Z}_6$ (with addition)

   (e) $\{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \ m \in \mathbb{Z}_3\}$

   (f) $\mathbb{Z}_2 \times \mathbb{Z}_3$

7. By analysing all possible Cayley tables show that, up to isomorphism, there is only one group of order 2, one group of order 3 and there are only two groups of order 4.