MA3615 Groups and Symmetry

Exercise Sheet 4: Normal subgroups, quotient groups, homomorphisms

1. Prove that $H = \langle(1,2,3)\rangle$ is a normal subgroup of S_3 but not a normal subgroup of S_4.

2. Prove that $N = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a normal subgroup of S_4. Construct the Cayley table for S_4/N.

3. Show that $\mathbb{Z}_8$ has a normal subgroup N isomorphic to $\mathbb{Z}_2$ with $\mathbb{Z}_8/N \cong \mathbb{Z}_4$.

4. Let G be the group of symmetries of the square. Let r^2 be the rotation of the square by π. Show that the subgroup $\langle r^2\rangle$ is a normal subgroup of G and that the quotient $G/\langle r^2\rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

5. There is a unique homomorphism $\theta : \mathbb{Z}_6 \to S_3$ such that $\theta(1) = (1,2,3)$. Find $\theta(n)$ for each $n \in \mathbb{Z}_6$. Determine the kernel and the image of θ.

6. Show that the map $\phi : S_n \to \mathbb{Z}_2$ defined by $\phi(\sigma) = 0$ if σ is an even permutation and $\phi(\sigma) = 1$ if σ is an odd permutation is a homomorphism. What is $\operatorname{Ker}\phi$? What can you deduce from the First Isomorphism theorem?