MA3615

CITY UNIVERSITY

London

BSc Honours Degree in Mathematical Science
Mathematical Science with Statistics
Mathematical Science with Computer Science
Mathematical Science with Finance and Economics
Mathematics and Finance

PART 3

Groups and Symmetry

2011

Time allowed: 2 hours

Full marks may be obtained for correct answers to THREE of the FOUR questions.
All necessary working must be shown.

Recall that for any positive integer n we denote by S_n the symmetric group of degree n, by $\mathbb{Z}_n$ the group $\mathbb{Z}_n = \{0, 1, , \ldots, n-1\}$ with addition modulo n, and by $C_n = \{e, r, r^2, \ldots, r^{n-1}\}$ the cyclic group of order n.

1. (a) Decide whether the following are groups. Justify your answers.

   i. $G = \{\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}\}$ with multiplication of matrices.

   ii. $H = \{1, 2\}$ with multiplication modulo 3.

   iii. $K = \{e, (1, 2), (1, 3), (2, 3)\}$ with composition of permutations.

   [6]

   (b) Explain, using a Cayley table, why there is only one group of order 3 up to isomorphism. **[8]**

   (c) i. Find two different isomorphisms from $\mathbb{Z}_3$ to C_3.

   ii. Find a homomorphism from $\mathbb{Z}_3$ to C_3 which is not an isomorphism.

   [3]

   (d) Classify the following three groups into isomorphism classes. For each pair of groups, if they are isomorphic find an explicit isomorphism between them, and if not then explain why not.

   i. $\mathbb{Z}_6$.

   ii. The group of all symmetries of an equilateral triangle.

   iii. S_3.

   [8]

2. (a) When do we say that a subgroup H of a group G is a normal subgroup? **[2]**

(b) Let H be a subgroup of a group G with $|G| = 2|H|$. Explain why H is normal in G. **[4]**

(c) Consider the dihedral group $D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$. (Recall that we have the following relations: $r^4 = e$, $s^2 = e$ and $sr^i = r^{4-i}s$ for $i = 1, 2, 3$.) Find the left and right cosets of the following subgroups of D_8.

i. $H_1 = \langle rs \rangle$.

ii. $H_2 = \langle r^2 \rangle$. **[6]**

Deduce that H_2 is a normal subgroup of D_8. Write down the Cayley table for D_8/H_2 and hence find an isomorphism between D_8/H_2 and $\mathbb{Z}_2 \times \mathbb{Z}_2$. **[6]**

(d) Recall that the group D_8 can be viewed as the group of all symmetries of the square (where r represents a rotation anticlockwise by $\frac{\pi}{2}$ and s represents a reflection through a vertical axis). By considering the action of D_8 on the two diagonals of the square, find a surjective homomorphism

$$\phi : D_8 \to S_2.$$

Hence, find a normal subgroup N of D_8 such that $D_8/N \cong S_2$. **[7]**

3. (a) Describe the group G consisting of all rotational symmetries of a regular tetrahedron. What is the order of G? **[4]**

(b) Suppose that the vertices of the tetrahedron are now coloured in some way and let G' be the group of all rotational symmetries of the coloured tetrahedron.

What is the relationship between G and G'?

Is it possible to colour the vertices of the tetrahedron in such way as to have

i. $|G'| = 3$? ii. $|G'| = 5$

If it is, give an example and if it is not, explain why not. **[6]**

(c) State Burnside's Counting Theorem. **[3]**

(d) How many different tetrahedrons can be constructed by colouring each vertex of a regular tetrahedron red, white or blue? Describe them all. **[12]**

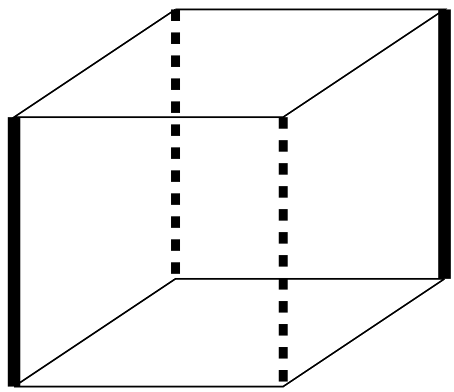
4. Let G be a group, X be a set and let $x \in X$.

   (a) Explain what is meant by 'G acts on X', 'the G-orbit of x' and 'the stabilizer G_x of x in G'. **[6]**

   (b) Prove that G_x is a subgroup of G. **[6]**

   (c) Suppose that G is finite group acting on a finite set X. State the Orbit-Stabilizer theorem. **[2]**

   (d) Now let G be the rotational symmetry group of the following painted cube.



   i. By considering the action of G on a suitable set, show that $|G| = 4$. **[4]**

   ii. State clearly the classification of all finite 3-dimensional rotation groups. **[2]**

   iii. Deduce from the classification that G is isomorphic to D_4. **[5]**

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