MA3615: Groups and Symmetry Mock Exam

(Full marks may be obtained for correct answers to three of the five questions. If more than three questions are answered, the best three marks will be credited.)

Recall that for any positive integer n we denote by S_n the symmetric group of degree n, by A_n the alternating group of degree n, and by $\mathbb{Z}_n$ the group $\mathbb{Z}_n = \{0, 1, , \ldots, n-1\}$ together with addition modulo n.

1. (a) Decide whether or not the following are groups. Justify your answers.
   i. $\mathbb{Z}$ with multiplication.
   ii. $\mathbb{Z}_4^* = \{1, 2, 3\}$ with multiplication modulo 4.
   iii. $\{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} : ab \neq 0,\ a, b, c \in \mathbb{R}\}$ with multiplication of matrices.

   (b) When do we say that two groups $(G, *_G)$ and $(H, *_H)$ are isomorphic?

   (c) Classify the following three groups into isomorphism classes. For each pair of groups, if they are isomorphic find an explicit isomorphism between them and if they are not, explain why not.
   i. $\mathbb{Z}_4$
   ii. All rotational symmetries of a 4-pyramid
   iii. $\mathbb{Z}_2 \times \mathbb{Z}_2$

2. (a) Let G be a group and let $g \in G$. Explain what is meant by the order of g in G.

   (b) Find the order of the following elements of S_5.
   i. $(1, 2, 3)(4, 5)$.
   ii. $(1, 2, 4, 5)$.

   (c) Let $D_6 = \{e, r, r^2, s, rs, r^2s\}$ be the dihedral group of order 6. (Recall that we have the following relations: $r^3 = e$, $s^2 = e$ and $sr^i = r^{3-i}s$ for $i = 1, 2$).
   i. Find the order of each element in D_6.
   ii. Suppose that $G = \{u, v, w, x, y, z\}$ is a group with operation $*$ where z has order 3, $y * z = u$ and $v * w = y$. Suppose further that we have an isomorphism

   $$\phi : D_6 \to G$$

   satisfying $\phi(s) = v$, $\phi(rs) = w$. Find the Cayley table for G.

Turn over ...

3. (a) When do we say that a subset H of a group G is a subgroup of G? Prove that $H_1 = \{e, (1,2,3), (1,3,2)\}$ and $H_2 = \{e, (1,2)\}$ are subgroups of S_3.

(b) When do we say that a subgroup is a normal subgroup? Explain why H_1 is a normal subgroup of S_3 but H_2 is not a normal subgroup of S_3. Are H_1 and H_2 normal subgroups of S_4?

(c) Find the cosets of H_1 in S_3 and write down the Cayley table for S_3/H_1. Find an isomorphism from S_3/H to $\mathbb{Z}_2$.

(d) Consider the map $\theta : S_n \to \mathbb{Z}_2$ given by

$$\theta(g) = \begin{cases} 0 & \text{if } g \text{ is an even permutation} \\ 1 & \text{if } g \text{ is an odd permutation} \end{cases}$$

Explain why θ is a surjective homomorphism and find $\operatorname{Ker}\theta$. Deduce that $S_n/A_n \cong \mathbb{Z}_2$.
(Note that $H_1 = A_3$ so this generalizes part (c)).

4. (a) Let G be a group acting on a set X and let $g \in G$. Define what is meant by $\mathrm{Fix}(g)$.

(b) State Burnside Counting Theorem.

(c) How many different tetrahedron can be constructed by painting each face of a regular tetrahedron red, white or blue?

Turn over ...

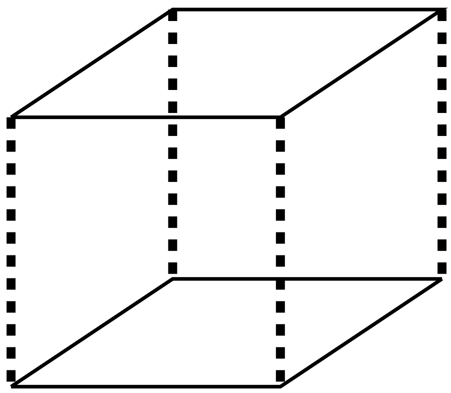
5. Let G be a group, X be a set and let $x \in X$.

(a) Explain what is meant by 'G acts on X', 'the G-orbit of x', 'the stabilizer G_x of x in G'.

(b) Prove that G_x is a subgroup of G.

(c) Suppose that G and X are both finite. State the Orbit-Stabilizer theorem.

(d) Now let G be the rotational symmetry group of the following painted cube.



i. By considering the action of G on a suitable set X show that $|G| = 8$.

ii. State the classification of all finite 3-dimensional rotation groups.

iii. Deduce from the classification that $G \cong D_8$.

Internal Examiner:	Dr M. De Visscher
External Examiners:	??
	??