Solutions to MA3615 Groups and Symmetry: January 2009 Exam

1. (a) **(4 marks)** A group is a set G together with an operation $*$ which assigns to each pair (g_1, g_2) of elements in G an element $g_1 * g_2$ in G satisfying the following conditions:
   (G1) There exists an element $e \in G$ such that $e * g = g * e = g$ for all $g \in G$.
   (G2) For all $g \in G$ there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.
   (G3) For all $g_1, g_2, g_3 \in G$ we have $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

   (b) **(2 marks)** $2 \times 3 = 0$ modulo 6 and $0 \notin A$.

   (c) **(4 marks)** Write the multiplication into the following table:

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

   So the multiplication is closed. Now check (G1)-(G3):
   (G1) identity $= 1$.
   (G2) $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$.
   (G3) follows from associativity for multiplication of integers.

   (d) **(1 marks)** Let r be the rotation anticlockwise by $\frac{\pi}{2}$ around the centre of the cube. Then $C = \{e, r, r^2, r^3\}$.

   (e) **(4 marks)** Note that $B = \langle 2 \rangle = \{1, 2, 2^2 = 4, 2^3 = 3\}$. Now consider the bijection $\phi : C \longrightarrow B$ given by $\phi(r^i) = 2^i$ for $i = 0, 1, 2, 3$. By definition, this map is an isomorphism. Alternatively, write down both Cayley tables, using the order of elements of B given above and check that these coincide.

   (f) **(3 marks)** In $\mathbb{Z}_2 \times \mathbb{Z}_2$ every element has order 2 (or 1 for the identity). But in B and C, there are two elements of order 4. Thus these groups cannot be isomorphic.

   (g) **(2 marks)** Take for example $\theta : B \to C$ defined by $\theta(g) = e$ for all $g \in B$. This is a homomorphism but not an isomorphism as it is not bijective.

2. (a) **(2 marks)** The order of an element $g \in G$ is the smallest positive integer $r \in \mathbb{N}$ satisfying $g^r = e$.

   (b) **(2 marks)** The order of any $g \in G$ divides the order of the group G.

   (c) **(4 marks)** Say that the order of g is equal to r. Then using (b) we know that $rs = 6$ for some s. So

$$g^7 = g^{6+1} = g^6 * g = g^{rs} * g = (g^r)^s * g = e^s * g = e * g = g.$$

   Thus if $g^7 = e$ then we have $g = e$.

(d) **(12 marks)** As G and S_3 are isomorphic, G must have one element of order 1 corresponding to e under ϕ, three elements of order 2 corresponding to $(1,2),(1,3),(2,3)$ under ϕ and two elements of order 3 corresponding to $(1,2,3)$ and $(1,3,2)$ under ϕ.

Using the fact that $\phi(1,2)=u$, $\phi(1,3)=v$ and w has order 2, we deduce that $\phi(2,3)=w$.

Now $u*v=\phi(1,2)*\phi(1,3)=\phi((1,2)(1,3))=\phi(1,3,2)=x$.

This leaves us with $\phi(1,2,3)=y$.

Now using the composition in S_3, we get the following Cayley table for G:

	e	u	v	w	y	x
e	e	u	v	w	y	x
u	u	e	x	y	w	v
v	v	y	e	x	u	w
w	w	x	y	e	v	u
y	y	v	w	u	x	e
x	x	w	u	v	e	y

3. (a) **(3 marks)** A subset H of a group G is a subgroup of G if and only if we have

(S1) $e\in H$.

(S2) For all $h_1,h_2\in H$ we have $h_1*h_2\in H$.

(S3) For all $h\in H$ we have $h^{-1}\in H$.

(b) **(7 marks)**

i. Let H be a subgroup of a finite group G. Then the order of H divides the order of G.

ii. The only divisors of 17 are 1 and 17. Thus, using Lagrange's theorem, we get that $\mathbb{Z}_{17}$ can only have subgroups of order 1 and 17. This means that the only subgroups are $\{0\}$ and $\mathbb{Z}_{17}$ itself.

iii. The only divisors of 9 are 1, 3 and 9. So, using Lagrange's theorem, we get that $\mathbb{Z}_9$ can only have subgroups of order 1,3 or 9. These are $\{0\}$, $\{0,3,6\}$ and $\mathbb{Z}_9$.

(c) **(10 marks)**

i. $H=\{0,3,6\}$ is a subgroup of $\mathbb{Z}_9$ of order 3. As $\mathbb{Z}_9$ is abelian, all subgroups are normal (as left and right cosets coincide).

ii. $H=\{0,3,6\}$, $1+H=\{1,4,7\}$, $2+H=\{2,5,8\}$.

The Cayley table for $\mathbb{Z}_9/H$ is given by

	H	$1+H$	$2+H$
H	H	$1+H$	$2+H$
$1+H$	$1+H$	$2+H$	H
$2+H$	$2+H$	H	$1+H$

iii. Consider the homomorphism

$$\phi\,:\,\mathbb{Z}_9\longrightarrow\mathbb{Z}_3\,:\,n\mapsto n\ \mathrm{modulo}\,3$$

mapping $n \in \mathbb{Z}_9$ to its residue modulo 3.
The kernel of ϕ is given by $\operatorname{Ker}\phi = \{0, 3, 6\} = H$. The image of ϕ is given by $\operatorname{Im}\phi = \mathbb{Z}_3$ (as $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(2) = 2$). Using the first isomorphism theorem we get that $\mathbb{Z}_9/\operatorname{Ker}\phi \cong \operatorname{Im}\phi$ and so $\mathbb{Z}_9/H \cong \mathbb{Z}_3$.

4. (a) **(8 marks)** Rotations by $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ around an axis passing through the centre of opposite faces. We have 3 such axes, so we get 9 rotations of this type.

   Rotations by $\frac{2\pi}{3}, \frac{4\pi}{3}$ around an axis passing through opposite vertices of the cube. We have 4 such axes, so we get 8 rotations of this type.

   Rotations by π around an axis passing through the middle of opposite edges. We have 6 such axis, so we get 6 rotations of this type.

   Adding the identity e, we get 24 rotational symmetries of the cube. So the order of G is 24.

   (b) **(12 marks)** Total number of painted cubes $= 3^6$. (3 possible colours for each face, 6 faces). Let G act on the set of all painted cubes. Then two painted cubes are considered to be different if they belong to different G-orbits. So the number of different painted cubes is equal to the number of G-orbits on X (which can be calculated using Burnside Counting theorem above).

   If r is a rotation by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ around an axis passing through the centre of opposite faces, then $Fix(r) = 3^3$ (and we have 6 such rotations).

   If s is a rotation by π around an axis passing through the centre of opposite faces, then $Fix(s) = 3^4$ (and we have 3 such rotations).

   If t is a rotation by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around an axis passing through opposite vertices of the cube, then $Fix(t) = 3^2$ (and we have 8 such rotations).

   If u is a rotation by π around an axis passing through the middle of opposite edges, then $Fix(u) = 3^3$ (and we have 6 such rotations).

   Thus we get that the number of different painted cubes is equal to

$$\frac{1}{24}(3^6 + (6 \times 3^3) + (3 \times 3^4) + (8 \times 3^2) + (6 \times 3^3)) = 57.$$

5. (a) **(2 marks)** Let G be a finite group acting on a set X and let $x \in X$. Denote by $Orb_G(x)$ the G-orbit of x and by G_x the stabilizer of x in G. Then we have

$$|G| = |Orb_G(x)|.|G_x|$$

   (b) **(4 marks)** Consider the action of G on the set X of black vertices of the cube. Then X consists of a single G-orbit. For $x \in X$, the stabilizer of x in G consists of the rotations around the main diagonal through that vertex x. So we get that $|G_x| = 3$. We deduce that

$$|G| = 4 \times 3 = 12.$$

   (c) **(3 marks)** Let G be a finite subgroup of $SO_3(\mathbb{R})$. Then G is isomorphic to one of the following groups:
   C_n $(n \geq 1)$, D_{2n} $(n \geq 2)$, A_4, S_4, A_5.

(d) **(3 marks)** As G has order 12, we must have $G \cong C_{12}$, D_{12} or A_4. Now as G is not abelian, it cannot be isomorphic to C_{12}. Moreover, as G does not contain any rotation of order 6, it cannot be isomorphic to D_{12}. Hence we must have $G \cong A_4$.

(e) **(8 marks)** Consider the action of G on the set X of black vertices of the cube. Numbering these vertices 1,2,3,4, this gives a homomorphism

$$\phi : G \longrightarrow S_4.$$

Note that if a rotation fixes all the black vertices then it must fix all the white vertices as well and so it must fix the cube. This shows that the kernel of ϕ is given by $\{e\}$ and so ϕ is one-to-one.

Now G consists of two types of rotations: rotations by π around an axis passing through the middle of opposite faces and rotations by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around a main diagonal of the cube. The first type of rotation gives (under ϕ) a permutation with cycle type $(a, b)(c, d)$ and the second gives a permutation of cycle type $(a)(b, c, d)$. In both cases we get an even permutation. So in fact we have

$$\phi : G \longrightarrow A_4.$$

Now as ϕ is one-to-one and $|G| = |A_4| = 12$ we get that in fact ϕ is also onto and thus it is a bijection. Hence this gives the required isomorphism from G to A_4.