Solutions to MA3615 Groups and Symmetry: May 2010 Exam

1. Let $G = \{e, x, y, z, u, v\}$ be a group with multiplication $*$ and identity element e. Suppose that G is abelian. Suppose further that $x^2 = y^2 = z$, $z^2 = v^2 = x$, $x * y = v$, $x * z = e$, $u * x = y$ and u has order 2.

   (a) Using the condition given and the fact that every element occurs precisely once in each row and column we get the following Cayley table for G.

	e	x	y	z	u	v
e	e	x	y	z	u	v
x	x	z	v	e	y	u
y	y	v	z	u	x	e
z	z	e	u	x	v	y
u	u	y	x	v	e	z
v	v	u	e	y	z	x

[6]

   (b) The order of a group is the cardinality of the group. The order of an element g in the group is the smallest positive integer r satisfying $g^r = e$. The order of any element divides the order of the group. **[3]**

   (c) $e^{-1} = e$, $x^{-1} = z$, $y^{-1} = v$, $z^{-1} = x$, $u^{-1} = u$, $v^{-1} = y$.
   e has order 1, u has order 2, x and z have order 3, y and v have order 6. **[2]**

   (d) i. G cannot be isomorphic to S_6. As $|S_6| = 6! \neq |G| = 6$, there cannot be a bijection from S_6 to G. **[1]**

   ii. G cannot be isomorphic to S_3 as G is abelian and S_3 is not (take for example $(1,2)(2,3) \neq (2,3)(1,2)$). **[2]**

   iii. G is isomorphic to $\mathbb{Z}_6$. As y has order 6 it generates G and so we can define an isomorphism $\phi : \mathbb{Z}_6 \to G$ by setting $\phi(1) = y$. This gives $\phi(0) = e$, $\phi(1) = y$, $\phi(2) = y^2 = z$, $\phi(3) = y^3 = u$, $\phi(4) = y^4 = x$ and $\phi(5) = y^5 = v$. **[3]**

   iv. $(1,1)$ is an element of order 6 in $\mathbb{Z}_2 \times \mathbb{Z}_3$, so we can define $\psi : \mathbb{Z}_2 \times \mathbb{Z}_3 \to G$ by setting $\psi(1,1) = y$. This gives $\psi(0,0) = e$, $\psi(1,1) = y$, $\psi(0,2) = z$, $\psi(1,0) = u$, $\psi(0,1) = x$ and $\psi(1,2) = v$. **[3]**

2. (a) A subset H of a group G is a subgroup of G if the following conditions are satisfied
   (S1) $e_G \in H$.
   (S2) For all $h_1, h_2 \in H$ we have $h_1 h_2 \in H$.
   (S3) For all $h \in H$ we have $h^{-1} \in H$. **[2]**

   (b) A subgroup H of G is a normal subgroup of G is $gH = Hg$ for all $g \in G$. **[2]**

   (c) i. $H = \langle (1,2) \rangle = \{e, (1,2)\}$.
   Left cosets:
   H
   $(1,3)H = \{(1,3), (1,2,3)\}$

$(2,3)H = \{(2,3),(1,3,2)\}$
Right cosets:
H
$H(1,3) = \{(1,3),(1,3,2)\}$
$H(2,3) = \{(2,3),(1,2,3)\}$.
As left and right cosets do not coincide, the subgroup H is not normal in G. **[3]**

ii. $H = \langle(1,2,3)\rangle = \{e,(1,2,3),(1,3,2)\}$.
Left/right cosets:
H
$(1,2)H = H(1,2) = \{(1,2),(1,3),(2,3)\}$.
As left and right cosets coincide, H is normal in G. **[2]**

iii. $H = \langle 10,15\rangle = 5\mathbb{Z}$. As $10n + 15m \in 5\mathbb{Z}$ we have $H \subseteq 5\mathbb{Z}$ but also $5 = 15 - 10 \in H$, so $5\mathbb{Z} \subseteq H$.
Left/right cosets:
$H = \{\ldots - 10, -5, 0, 5, 10, \ldots\}$
$1 + H = \{\ldots - 9, -4, 1, 6, 11, \ldots\}$
$2 + H = \{\ldots - 8, -3, 2, 7, 12, \ldots\}$
$3 + H = \{\ldots - 7, -2, 3, 8, 13, \ldots\}$
$4 + H = \{\ldots - 6, -1, 4, 9, 14, \ldots\}$
As left and right cosets coincide ($\mathbb{Z}$ is abelian) the subgroup H is normal in $\mathbb{Z}$. **[3]**

(d) First consider $G = S_3$ and $H = \{e,(1,2,3),(1,3,2)\}$ then the Cayley table for G/H is given by

	H	$(1,2)H$
H	H	$(1,2)H$
$(1,2)H$	$(1,2)H$	H

We can see from the table (or otherwise) that $G/H \cong S_2$ ($\cong \mathbb{Z}_2 \cong C_2$). **[3]**
Next consider $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$. Then the Cayley table for G/H is given by

	H	$1+H$	$2+H$	$3+H$	$4+H$
H	H	$1+H$	$2+H$	$3+H$	$4+H$
$1+H$	$1+H$	$2+H$	$3+H$	$4+H$	H
$2+H$	$2+H$	$3+H$	$4+H$	H	$1+H$
$3+H$	$3+H$	$4+H$	H	$1+H$	$2+H$
$4+H$	$4+H$	H	$1+H$	$2+H$	$3+H$

We see from the Cayley table that $G/H \cong \mathbb{Z}_5$. **[5]**

3. (a) Let G be a finite group acting on a finite set X. For $g \in G$, define $\mathrm{Fix}(g)$ to be

$$\mathrm{Fix}(g) = |\{x \in X \mid g(x) = x\}.$$

[2]

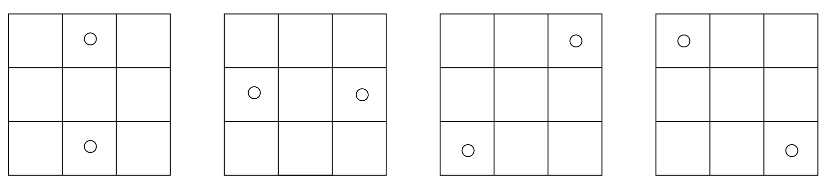
(b) Let G be a finite group acting on a finite set X then the number of G-orbits on X is given by

$$\frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g).$$

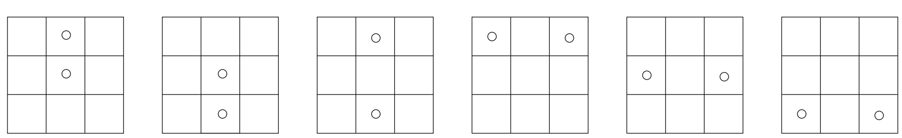
[2]

(c) i. Let $G = D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$ be the group of all symmetries of a square. Let X be the set of all punched cards. Then $|X| = \binom{9}{2} = 36$ (choose to punch 2 out of the 9 small squares). Then G acts on X and two cards will be the same precisely if they are in the same G-orbit. So we need to count the number of G-orbits on X. This is given by Burnside Counting theorem. **[4]**

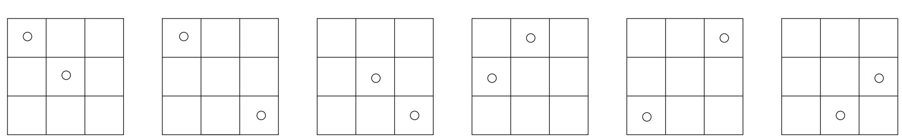
We have to find $\mathrm{Fix}(g)$ for each $g \in G$.
$\mathrm{Fix}(e) = 36$ as e fixes everything.
$\mathrm{Fix}(r) = \mathrm{Fix}(r^3) = 0$ as r and r^3 don't fix anything.
$\mathrm{Fix}(r^2) = 4$ as r^2 fixes the following 4 cards.



$\mathrm{Fix}(s) = \mathrm{Fix}(r^2s) = 6$ as s fixes the following 6 cards. (similarly for r^2s).



$\mathrm{Fix}(rs) = \mathrm{Fix}(r^3s) = 6$ as rs fixes the following 6 cards. (similarly for r^3s).

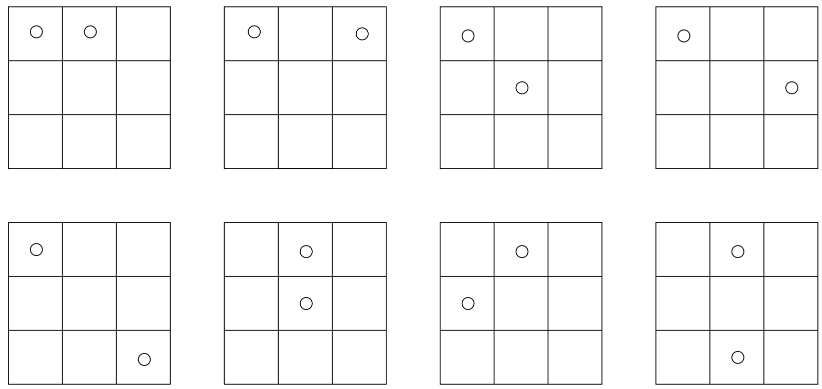


So applying Burnside theorem we get that the number of different ID cards is given by

$$\frac{1}{8}(36 + 0 + 0 + 4 + 6 + 6 + 6 + 6) = \frac{1}{8}(64) = 8.$$

[8]

ii. The 8 different ID cards are pictured below.



[4]

4. (a) Let G be a finite group acting on a finite set X and let $x \in X$. Denote by $\mathrm{Orb}_G(x)$ the G-orbit of x and by G_x the stabilizer of x in G. Then we have

$$|G| = |\mathrm{Orb}_G(x)|.|G_x|.$$

[2]

(b) Let G be the group of all rotational symmetries of a cube. Let X be the set of faces of the cube and let $x \in X$ be any face. Then $|Orb_G(x)| = 6$ and $|G_x| = 4$ (4 rotations around an axis passing through the centre of x). So applying the Orbit-Stabilizer theorem we get that $|G| = 4 \times 6 = 24$. **[2]**

(c) G' is a subgroup of G. **[2]**

i. No, as $|G| = 24$ and using Lagrange's theorem, $|G'|$ must divide 24. **[2]**

ii. Yes. We can for example paint the cube as a die (numbering the faces with the numbers $1, \ldots 6$). **[2]**

(d) i. Let x be any face of the painted cube. As G' acts transitively on the set of faces, we have that $|Orb_{G'}(x)| = 6$. Now $|G'_x| = 2$ as G'_x consists of the identity and a rotation by π. Thus using the Orbit-Stabilizer theorem we have $|G'| = 2 \times 6 = 12$. **[2]**

ii. Let G be any finite subgroup of $SO_3(\mathbb{R})$ then G is isomorphic to precisely one of the following: C_n $(n \geq 1)$, D_{2n} $(n \geq 2)$, A_4, S_4, A_5. **[2]**

iii. As $|G'| = 12$ we have that $G' \cong C_{12}$, D_{12} or A_4. Now as G' is not abelian we know that G' cannot be isomorphic to C_{12}. Morover as G' doesn't contain a rotation of order 6 we know that G' cannot be isomorphic to D_{12}. Hence, the group G' must be isomorphic to A_4. **[6]**