MA3615 Groups and Symmetry: Solutions to Coursework 1

1. (a) A is not a group as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse.

   (b) B is a group. $3n + 3m = 3(n+m) \in 3\mathbb{Z}$. Moreover we have
   (G1) identity $0 = 3.0 \in 3\mathbb{Z}$.
   (G2) inverse $(3n)^{-1} = -3n = 3(-n) \in 3\mathbb{Z}$.
   (G3) $3n + (3m + 3l) = (3n + 3m) + 3l$.

   (c) C is a group. In fact C is isomorphic to the group C_4 of all rotational symmetries of the square (to see this label the vertices of the square clockwise by 1,2,3 and 4).
   Can check that C is a group directly: First check that C is closed under multiplication:
   $e^2 = e \in C$, $(1,2,3,4)^2 = (1,3)(2,4) \in C$, $((1,3)(2,4))^2 = e \in C$, $(1,4,3,2)^2 = (1,3)(2,4) \in C$.
   $(1,2,3,4) \circ (1,3)(2,4) = (1,3)(2,4) \circ (1,2,3,4) = (1,4,3,2) \in C$.
   $(1,2,3,4) \circ (1,4,3,2) = (1,4,3,2) \circ (1,2,3,4) = e \in C$.
   $(1,3)(2,4) \circ (1,4,3,2) = (1,4,3,2) \circ (1,3)(2,4) = (1,2,3,4) \in C$. Moreover we have
   (G1) identity e.
   (G2) $e^{-1} = e$, $(1,2,3,4)^{-1} = (1,4,3,2)$, $((1,3)(2,4))^{-1} = (1,3)(2,4)$.
   (G3) follows from associativity of composition of maps.

   B and C are both abelian.

2. (a) Let r be the rotation by $\frac{\pi}{2}$ around the axis passing through the top vertex and the centre of the square base of the pyramid. Then $G = \{e, r, r^2, r^3\}$.

   (b) The map $\phi : G \to H$ given by $\phi(e) = 1$, $\phi(r) = 2$, $\phi(r^2) = 4$ and $\phi(r^3) = 3$ is an isomorphism. (Note that this map is completely determined once we set $\phi(r) = 2$. We could also have chosen $\phi(r) = 3$ as 3 is also an element of order 4.)

   (c) Label the vertices of the square base of the pyramid by 1,2,3 and 4. This defines an isomorphism $\theta : G \to C$ given by $\theta(e) = e$, $\theta(r) = (1,2,3,4)$, $\theta(r^2) = (1,3)(2,4)$ and $\theta(r^3) = (1,4,3,2)$.

   (d) No as in $\mathbb{Z}_2 \times \mathbb{Z}_2$ every element (other than $(0,0)$) as order 2, but in G we have two elements of order 4, namely r and r^3.

3. (a) As $|G| = 6$ we have that $g^6 = e$.
   Now $g^{13} = g^6 * g^6 * g = e * e * g = g$. So if $g^{13} = e$ then we must have $g = e$.

   (b) As $|G| = 17$ is prime, using Lagrange's theorem we know that subgroups of G can only have order 1 or 17. So the only subgroups of G are $\{e\}$ and G itself.

4. $(G, \dagger)$ cannot be a group as the last row of the Cayley table contains the entry a twice. This contradicts Corollary 1.5 from the lecture which says that in the Cayley table of a finite group every element appears precisely once in each row and column.

$(G, *)$ is a group as we have an isomorphism with $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by (for example) $\phi : G \to \mathbb{Z}_2 \times \mathbb{Z}_2$ with $\phi(a) = (1,0)$, $\phi(b) = (0,0)$, $\phi(c) = (0,1)$ and $\phi(d) = (1,1)$.
Note that the Cayley table for $(G, *)$ forces b to be the identity, and so we must have $\phi(b) = (0,0)$, but in fact a, c, d could be mapped to anything, it would always give an isomorphism.

$(G, \times)$ is also a group as we have an isomorphism $\psi : G \to \mathbb{Z}_4$ with $\psi(a) = 1$, $\psi(b) = 2$, $\psi(c) = 0$ and $\psi(d) = 3$.
Here again, the Cayley table for $(G, \times)$ forces c to be the identity. Now we also see from the Cayley table that G has two elements of order 4, namely a and d, and one element of order 2, namely b. So a could be mapped to 1 or 3 in $\mathbb{Z}_4$. If we say that $\psi(a) = 1$ then as ψ is an isomorphism, everything else is automatically determined as $\psi(b) = \psi(a^2) = \psi(a)^2 = 1 + 1 = 2$, $\psi(d) = \psi(a^3) = \psi(a)^3 = 1 + 1 + 1 = 3$.

5. Note that the assumptions given and the fact that ϕ is an isomorphism determine ϕ completely:
$g^2 = i$ implies that $\phi(1,2)^2 = \phi((1,2)^2) = \phi(e) = i$.
$g * h = j$ implies that $\phi(1,2) * \phi(2,3) = \phi((1,2)(2,3)) = \phi(1,2,3) = j$.
As k has order 3, it must be the image of an element of order 3, but there is only one such element left, namely $(1,3,2)$. Thus we must have $\phi(1,3,2) = k$.
This leaves $\phi(1,3) = l$. Now using the multiplication in S_3 and the isomorphism ϕ we get that the Cayley table for $(K, *)$ is given by

	i	g	h	l	k	j
i	i	g	h	l	k	j
g	g	i	j	k	l	h
h	h	k	i	j	g	l
l	l	j	k	i	h	g
k	k	h	l	g	j	i
j	j	l	g	h	i	k

6. Subgroup of order 1: $\{0\}$.
Subgroup of order 2: $\{0, 6\}$.
Subgroup of order 3: $\{0, 4, 8\}$.
Subgroup of order 4: $\{0, 3, 6, 9\}$.
Subgroup of order 6: $\{0, 2, 4, 6, 8, 10\}$.
Subgroup of order 12: $\mathbb{Z}_{12}$.

7. (a) $H = \langle r^3 \rangle = \{e, r, r^2, r^3\}$
$sH = Hs = \{s, rs, r^2s, r^3s\}$.
   (b) $H = \langle r^2 s \rangle = \{e, r^2 s\}$
The left cosets are H, $rH = \{r, r^3 s\}$, $r^2 H = \{r^2, s\}$ and $r^3 H = \{r^3, rs\}$.
The right cosets are H, $Hr = \{r, rs\}$, $Hr^2 = \{r^2, s\}$ and $Hr^3 = \{r^3, r^3 s\}$.
   (c) $H = \langle r^2, s \rangle = \{e, r^2, s, r^2 s\}$
$rH = Hr = \{r, r^3, rs, r^3 s\}$.
   (d) $H = \langle r^3, s \rangle = D_8$ (only one coset).