MA3615 Groups and Symmetry: Solutions to Coursework

1. (a) [**5**] There exists a subgroup H of A_5 with $|H| = 9$.
This is false. We have $|A_5| = 60$, so by Lagrange's theorem the order of every subgroup of A_5 must divide 60. As 9 does not divide 60, we have that A_5 cannot have a subgroup of order 9.

   (b) [**5**] Every element $g \in S_4$ satisfies $g^{48} = e$.
This is true. As $|S_4| = 24$, we know that $g^{24} = e$ for every $g \in S_{24}$. Thus $g^{48} = (g^{24})^2 = e^2 = e$ for every $g \in S_4$.

   (c) [**5**] The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by one element.
This is false. In $\mathbb{Z}_2 \times \mathbb{Z}_2$ every element has order 2 or 1. Thus this group cannot be generated by one element.

   (d) [**5**] The group $\mathbb{Z}_2 \times \mathbb{Z}_3$ is generated by one element.
This is true. This group is generated by $(1, 1)$.

   (e) [**5**] The group D_6 has a proper non-trivial normal subgroup.
This is true. Take $N = \{e, r, r^2\}$. This is clearly a subgroup and as $|N| = 3 = \frac{|D_8|}{2}$ we have that N is a normal subgroup.

   (f) [**5**] The group D_6 has a subgroup which is not normal.
This is true. Take for example $H = \{e, s\}$. This is clearly a subgroup but it is not normal as $rH = \{r, rs\} \neq Hr = \{r, sr = r^2 s\}$.

2. (a) [**5**] $H = \{0, 5\}$. (Check (S1)-(S3)).

   (b) [**5**] As $\mathbb{Z}_{10}$ is abelian, every subgroup is a normal subgroup (left and right cosets coincide).

   (c) [**5**] $H = \{0, 5\}$, $1 + H = \{1, 6\}$, $2 + H = \{2, 7\}$, $3 + H = \{3, 8\}$, $4 + H = \{4, 9\}$.

	H	$1+H$	$2+H$	$3+H$	$4+H$
H	H	$1+H$	$2+H$	$3+H$	$4+H$
$1+H$	$1+H$	$2+H$	$3+H$	$4+H$	H
$2+H$	$2+H$	$3+H$	$4+H$	H	$1+H$
$3+H$	$3+H$	$4+H$	H	$1+H$	$2+H$
$4+H$	$4+H$	H	$1+H$	$2+H$	$3+H$

   (d) [**5**] Define $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_5$ sending $n \in \mathbb{Z}_{10}$ to its residue modulo 5. Then ϕ is a homomorphism. Moreover, as $\phi(i) = i$ for all $i = 0, 1, 2, 3, 4$ we have that ϕ is surjective. Now $\operatorname{Ker} \phi = \{0, 5\} = H$. So using the first isomorphism theorem we have
$$\mathbb{Z}_{10}/H \cong \operatorname{Im} \phi = \mathbb{Z}_5.$$

3. (a) [**10**] We claim that G acts transitively on the set of faces of the cube. To see this, start with the top face of the cube. Then, by applying the rotations around an axis through the centre of the front and back faces, we can obtain the side

faces and the bottom face. And, by applying rotations around an axis through the centre of the side faces, we can also obtain the front and back faces.
This means that there is only one G-orbit on the set X. Let $x \in X$ then we have $Orb_G(x) = X$ and

$$|Orb_G(x)| = 6.$$

Now the stabilizer of x consists of the identity and the rotations by $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ around the axis passing through the centre of x and the centre of the opposite face. So $|G_x| = 4$. Thus we get

$$|G| = |Orb_G(x)|.|G_x| = 6 \times 4 = 24.$$

(b) **[10]** We claim that G acts transitively on the set of edges of the cube. To see this, first note that starting from an edge x we always have that all the other edges on the same face as x are in the same orbit as x (by applying the rotations around an axis passing through the centre of that face). But now, using the argument in part (a), we have that all the faces are in the same orbit. So we can deduce that all the edges are in the same orbit.
So, for any $x \in X$ we have $Orb_G(x) = X$ and

$$|Orb_G(x)| = 12.$$

Now the stabilizer of x consists of the identity and the rotation by π around the axis passing through the middle of x and the middle of the opposite edge. So $|G_x| = 2$. Thus we get

$$|G| = |Orb_G(x)|.|G_x| = 12 \times 2 = 24.$$

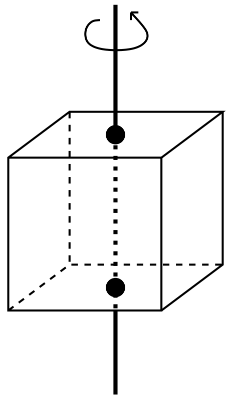
4. (a) **[15]** Consider the set X of all painted cubes obtained by painting each edge of a cube red or blue (without moving the cube). There are two possible colours for each edge, so we get $|X| = 2^{12}$.
The group G of all rotational symmetries of the cube acts on the set X. Painted cubes are considered to be different if they are in different G-orbits. Thus the number of different painted cubes is equal to the number of G-orbits on X. We use Burnside's Counting theorem to count the number of G-orbits. It is given by

$$\frac{1}{|G|}\sum_{g \in G} \text{Fix}(g)$$

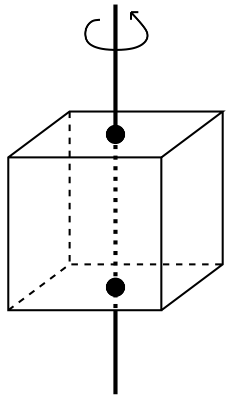
where $\text{Fix}(g) = |\{x \in X \mid g(x) = x\}|$.

Now we have $\text{Fix}(e) = 2^{12}$.

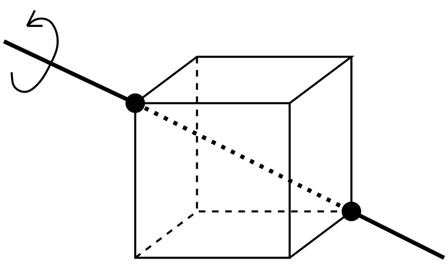
Let r_1 be a rotation by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ around an axis passing through the centre of a face and the centre of the opposite face. Then $\text{Fix}(r_1) = 2^3$ as the top four edges have to be painted in the same colour, the bottom four edges have to be painted in the same colour and the side four edges have to be painted in the same colour. Note that there are 6 such rotations.



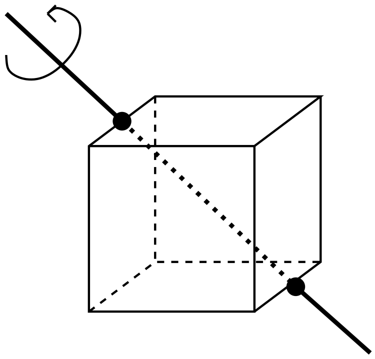
Let r_2 be a rotation by π around an axis through the centre of a face and the centre of the opposite face. Then $\mathrm{Fix}(r_2) = 2^6$ as the top four edges can be painted in two (possibly different) colours, and similarly for the bottom and the side edges. Note that there are 3 such rotations.



Let r_3 be a rotation by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around a main diagonal of the cube (from vertex v to vertex v'). Then $\mathrm{Fix}(r_3) = 2^4$ as the 3 edges coming out of v have to be painted in the same colour, the 3 edges coming out of v' must be painted in the same colour and the remaining 6 edges can be painted in 2 (possibly different) colours. Note that there are 8 such rotations.



Let r_4 be a rotation by π around an axis through the middle of an edge and the middle of the opposite edge. Then $\mathrm{Fix}(r_4) = 2^7$ as the edges intersecting the axis of rotation can be painted in any colour and the remaining 10 edges have to be coloured in pairs. Note that there are 6 such rotations.



Putting everything together we get that the number of different painted cubes is equal to

$$\frac{1}{24}(2^{12} + (2^3 \times 6) + (2^6 \times 3) + (2^4 \times 8) + (2^7 \times 6) = 218.$$

(b) **[15]** The argument is similar as the one used for part (a). Here we denote by X the set of all cubes obtained by painting each face red, yellow or blue. We have $|X| = 3^6$. The group G is as in part (a) and we use the same notation to denote the rotations in G. Considering $\mathrm{Fix}(g)$ in each case we get $\mathrm{Fix}(e) = 3^6$, $\mathrm{Fix}(r_1) = 3^3$, $\mathrm{Fix}(r_2) = 3^4$, $\mathrm{Fix}(r_3) = 3^2$, $\mathrm{Fix}(r_4) = 3^3$. So we get that the total number of painted cube is given by

$$\frac{1}{24}(3^6 + (3^3 \times 6) + (3^4 \times 3) + (3^2 \times 8) + (3^3 \times 6)) = 57.$$