MA3615 Groups and Symmetry

Solutions to Exercise Sheet 1

1. e : identity.
   r : rotation anticlockwise by $\frac{\pi}{2}$.
   r^2 : rotation anticlockwise by π.
   r^3 : rotation anticlockwise by $\frac{3\pi}{2}$.
   $s_1 = s$: reflection through vertical line.
   $s_2 = rs$: reflection through NW-SE line.
   $s_3 = r^2s$: reflection through horizontal line.
   $s_4 = r^3s$: reflection through NE-SW line.

	e	r	r^2	r^3	s	rs	r^2s	r^3s
e	e	r	r^2	r^3	s	rs	r^2s	r^3s
r	r	r^2	r^3	e	rs	r^2s	r^3s	s
r^2	r^2	r^3	e	r	r^2s	r^3s	s	rs
r^3	r^3	e	r	r^2	r^3s	s	rs	r^2s
s	s	r^3s	r^2s	rs	e	r^3	r^2	r
rs	rs	s	r^3s	r^2s	r	e	r^3	r^2
r^2s	r^2s	rs	s	r^3s	r^2	r	e	r^3
r^3s	r^3s	r^2s	rs	s	r^3	r^2	r	e

   order of G, $|G| = 8$.
   G is not abelian as $sr = r^3s \neq rs$.
   $e^{-1} = e$, $r^{-1} = r^3$, $(r^2)^{-1} = r^2$, $s^{-1} = s$, $(rs)^{-1} = rs$, $(r^2s)^{-1} = r^2s$, $(r^3s)^{-1} = r^3s$.

2. (a) Not a group as (G2) fails (2 has no inverse).

   (b) $(2\mathbb{Z}, +)$ is a group as for all $2n, 2m \in 2\mathbb{Z}$ we have $2n + 2m = 2(n+m) \in 2\mathbb{Z}$. Moreover we have
   (G1) $e = 0 \in 2\mathbb{Z}$
   (G2) For all $2n \in 2\mathbb{Z}$ we have $(2n)^{-1} = -2n = 2(-n) \in 2\mathbb{Z}$.
   (G3) $2n + (2m + 2p) = (2n + 2m) + 2p$ for all $2n, 2m, 2p \in 2\mathbb{Z}$.

   (c) Not a group as $3 + 5 = 8$ which is not an odd integer.

   (d) Not a group as 0 has no inverse (so (G2) fails).

   (e) It is a group as $\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd} \in \mathbb{Q}^+$ and we have
   (G1) $e = 1$.
   (G2) For all $\frac{a}{b} \in \mathbb{Q}^+$ we have $(\frac{a}{b})^{-1} = \frac{b}{a} \in \mathbb{Q}^+$.
   (G3) obvious.

   (f) It is a group. This set is closed under multiplication modulo 5. Moreover we have
   (G1) $e = 1$.
   (G2) $1^{-1} = 1$, $2^{-1} = 3$ (as $2.3 = 6 = 1$ modulo 5), $3^{-1} = 2$, $4^{-1} = 4$.
   (G3) $a.(b.c) = (a.b).c$ (still true modulo 5).

(g) Not a group. (G2) fails as 3 has no inverse.

(h) It is a group. This set of vectors is closed under addition of vectors modulo 2. Moreover we have
(G1) $e = (0,0)$.
(G2) $(0,0)^{-1} = (0,0)$, $(0,1)^{-1} = (0,1)$, $(1,0)^{-1} = (1,0)$, $(1,1)^{-1} = (1,1)$.
(G3) $(a,b) + ((c,d) + (e,f)) = (a+c+e, b+d+f) = ((a,b) + (c,d)) + (e,f)$.

(i) It is a group. We have

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix} \in G$$

Moreover we have
(G1) $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$.
(G2) $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \in G$.
(G3) follows from associativity of matrix multiplication (or addition in $\mathbb{Z}_5$).

3. (b) abelian, order is infinite.
(e) abelian, order is infinite.
(f) abelian, order is 4.
(h) abelian, order is 4.
(i) abelian, order is 5.

4. (a)

$$\begin{aligned}(x*y)*(y^{-1}*x^{-1}) &= x*(y*y^{-1})*x^{-1} \qquad \text{using (G3)}\\ &= x*e*x^{-1} \qquad \text{using (G2)}\\ &= x*x^{-1} \qquad \text{using (G1)}\\ &= e \qquad \text{using (G2).}\end{aligned}$$

Similarly we have $(y^{-1}*x^{-1})*(x*y) = e$. Thus by definition of the inverse we have that $(x*y)^{-1} = y^{-1}*x^{-1}$.

(b) Starting from $x*y = e$ and multiplying by x on the right we get

$$x*y*x = e*x$$

Using (G1) we get $x*y*x = x$ and so using (G1) again we get $x*y*x = x*e$. Thus using the Cancellation rule (Theorem 1.2 from the lecture) we see that $y*x = e$ as required.

(c) Assume $(x*y)^2 = x^2*y^2$. Expanding we get

$$x*y*x*y = x*x*y*y.$$

Now using the cancellation rule twice we get

$$y*x = x*y$$

as required.

5.

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The first row and column follow from the fact that e is the identity. The main diagonal is given in the question. Now $a * b$ cannot be e, a or b (as they all appear already in the same row or column) so it has to be c. The rest of the table follows by similar arguments.

6. If this was the Cayley table of a group, then e would have to be the identity. Now $a * b = e$ but $b * a = c \neq e$. This contradicts question 4(b) above.