MA3615 Groups and Symmetry

Solutions to Exercise Sheet 2

1. (a) $\alpha = (1,4,3,2)(5,7,6)$, $\beta = (1,2,6)(3,4)(5)(6) = (1,2,6)(3,4)$.

   (b) $\alpha \circ \beta = (1)(2,5,7,6,4)(3)$, $\beta \circ \alpha = (1,3,6,5,7)(2)(4)$ and $\alpha^2 = (1,3)(2,4)(5,6,7)$

   (c) $\alpha = (3,2)(4,2)(1,2)(7,6)(5,6)$ is an odd permutation. $\beta = (2,6)(1,6)(3,4)$ is also an odd permutation.

2.
$$(x_1, x_2, \ldots x_n) = (x_{n-1}, x_n)(x_{n-2}, x_n) \ldots (x_2, x_n)(x_1, x_n)$$
   can be written as a product of $n-1$ transpositions. So $(x_1, x_2, \ldots x_n)$ is an even permutation if and only if n is odd.

   (a) $(1,2)$ is odd, $(3,6,8)$ is even and $(4,11,10,5,9,7)$ is odd. So $(1,2)(3,6,8)(4,11,10,5,9,7)$ is even (odd + even + odd = even).

   (b) $(1,3,5,7,9,11,2,4,6,8)$ is odd.

   (c) $(1,2,3,4)(1,2,4,3)(1,4,2,3)$ is odd (=odd + odd + odd).

3. Clearly we have
$$e \circ g = g \circ e \in V_4 \qquad \forall g \in V_4.$$
   Now $((1,2)(3,4))^2 = ((1,3)(2,4))^2 = ((1,4)(2,3))^2 = e \in V^4$. Also we have
   $(1,2)(3,4) \circ (1,3)(2,4) = (1,3)(2,4) \circ (1,2)(3,4) = (1,4)(2,3) \in V_4$
   $(1,2)(3,4) \circ (1,4)(2,3) = (1,4)(2,3) \circ (1,2)(3,4) = (1,3)(2,4) \in V_4$
   $(1,3)(2,4) \circ (1,4)(2,3) = (1,4)(2,3) \circ (1,3)(2,4) = (1,2)(3,4) \in V_4$.
   So the multiplication is closed.
   Now we have
   (G1) $e \in V_4$.
   (G2) $g^{-1} = g \in V^4$ for all $g \in V_4$.
   (G3) follows from associativity of composition of permutations.

4. $x * w = \theta(1,2) * \theta(1,3,2) = \theta((1,2)(1,3,2)) = \theta(1,3) = y$,
   $w^{-1} = (\theta(1,3,2))^{-1} = \theta((1,3,2)^{-1}) = \theta(1,2,3) = v$,
   $v^5 = (\theta(1,2,3))^5 = \theta((1,2,3)^5) = \theta(1,3,2) = w$,
$$\begin{aligned} z * v^{-1} * x &= \theta(2,3) * (\theta(1,2,3))^{-1} * \theta(1,2) \\ &= \theta(2,3) * \theta((1,2,3)^{-1}) * \theta(1,2) \\ &= \theta((2,3)(1,3,2)(1,2)) \\ &= \theta(e) \\ &= u. \end{aligned}$$

5. Let G be the group of symmetries of a (non-square) rectangle. Write $G = \{e, s, t, r\}$ where s denotes the reflection through the vertical line, t denotes the reflection through the horizontal line and r denotes the rotation by π around the centre of the rectangle.

   Label the vertices of the rectangle clockwise using the numbers 1,2,3,4 where 1 is the NW vertex. Then each symmetry of the rectangle gives rise to a permutation on $\{1, 2, 3, 4\}$ and so we get a map $\theta : G \to V_4$ given by $\theta(e) = e$, $\theta(s) = (1, 2)(3, 4)$, $\theta(t) = (1, 4)(2, 3)$ and $\theta(r) = (1, 3)(2, 4)$. It is easy to check that this gives the required isomorphism.

   In Exercise Sheet 1, question 2(h) we have the group

$$H = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

   with addition of vectors modulo 2. The following map is an example of an isomorphism between H and V_4.

$$\phi : H \to V_4, \qquad \begin{aligned} \phi(0, 0) &= e \\ \phi(0, 1) &= (1, 2)(3, 4) \\ \phi(1, 0) &= (1, 4)(2, 3) \\ \phi(1, 1) &= (1, 3)(2, 4). \end{aligned}$$

6. Write

   G_1 = the group of rotational symmetries of an equilateral triangle.
   G_2 = the group of all symmetries of an equilateral triangle.
   $G_3 = S_3$.
   $G_4 = \mathbb{Z}_6$ (with addition).
   $G_5 = \{\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \ m \in \mathbb{Z}_3\}$ with multiplication of matrices.
   $G_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$.

   Then G_1 and G_5 have order 3 and all the other groups have order 6. Thus G_1 and G_5 cannot be isomorphic to any other group.

   We claim that G_1 and G_5 are isomorphic. Denote be r the rotation of the triangle anticlockwise by $\frac{2\pi}{3}$. Then $G_1 = \{e, r, r^2\}$. Now define a map $\theta : G_1 \to G_5$ by $\theta(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\theta(r) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\theta(r^2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. It is easy to check that this gives a isomorphism between G_1 and G_5.

   Among the groups of order 6, we have that G_4 and G_6 are abelian but G_2 and G_3 are not. So the first two cannot be isomorphic to any of the last two.

   We claim that $G_4 \cong G_6$. The group $G_4 = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 and the group $G_6 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ with addtion of vectors module 2 for the first coordinate and modulo 3 for the second coordinate. Define $\theta : G_4 \to G_6$ by setting $\theta(1) = (1, 1)$. Note that this completely determine the isomorphism θ as $\theta(2) = \theta(1 + 1) = \theta(1) + \theta(1) = (1, 1) + (1, 1) = (0, 2)$, $\theta(3) = \theta(2 + 1) = (1, 0)$, $\theta(4) = \theta(3 + 1) = (0, 1)$ and $\theta(5) = \theta(4 + 1) = (1, 2)$.

Finally we claim that $G_2 \cong G_3$. To see this, write $G_2 = \{e, r, r^2, s, rs, r^2s\}$ where r denotes the rotation anticlockwise by $\frac{2\pi}{3}$ and s denotes the reflection through the vertical line. If we label the vertices of the triangle anticlockwise by 1,2,3 with 1 at the top, then each symmetry of the triangle gives rise to a permutation of $\{1, 2, 3\}$ and so we get an isomorphism $\theta : G_2 \to G_3$ such that $\theta(e) = e$, $\theta(r) = (1, 2, 3)$, $\theta(r^2) = (1, 3, 2)$, $\theta(s) = (2, 3)$, $\theta(rs) = (1, 2)$, $\theta(r^2s) = (1, 3)$.

Thus we get three isomorphism classes, $\{G_1, G_5\}$, $\{G_2, G_3\}$ and $\{G_4, G_6\}$.

7. Let G be a group of order 2, then $G = \{e, g\}$. So its Cayley table starts as

$$\begin{array}{c|cc} * & e & g \\ \hline e & e & g \\ g & g & \end{array}$$

There is only one way of completing this table, namely as

$$\begin{array}{c|cc} * & e & g \\ \hline e & e & g \\ g & g & e \end{array}$$

Thus, up to isomorphism, there is only one possible group of order 2. (In fact it is easy to see that $G \cong C_2$ the cyclic group of order 2).

Let $G = \{e, a, b\}$ be a group of order 3. Then its Cayley table starts as

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & & \\ b & b & & \end{array}$$

Now either $a^2 = e$ or $a^2 = b$. But if $a^2 = e$ then we must have $a * b = b$ and so, by uniqueness of the identity, a would have to be the identity which is a contradiction. Thus we must have $a^2 = b$. Once this is established, there is only one way to complete the rest of the Cayley table and we get

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array}$$

Thus, up to isomorphism, there is only one possible group of order 3. (In fact it is easy to see that $G \cong C_3$ the cyclic group of order 3).

Let $G = \{e, a, b, c\}$ be a group of order 4 where e is the identity. So the Cayley table starts as

$$\begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & & & \\ b & b & & & \\ c & c & & & \end{array}$$

We claim that at least one of a, b or c squares to e. Suppose, for a contradiction that this is not the case, i.e. that e does not appear on the diagonal except in the 1,1 position. Now e must appear in the second row, thus either $a * b = e$ or $a * c = e$. We can assume without loss of generality (by swapping b and c if necessary) that $a * b = e$. Now as e must also appear in the last column we must have $b * c = e$, but then we cannot have $b * a = e$ as this would produce two e's in the same row. So we would have $a * b = e$ and $b * a \neq e$ which is a contradiction.

We can assume, without loss of generality (relabelling if necessary), that $a^2 = e$. Now, as a is not the identity, we have to complete the table as follows

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c		
c	c	b		

Now there are two ways of completing this table, giving rise to two non-isomorphic groups. The first type has Cayley table given by

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This group is in fact isomorphic to V_4 (find an explicit isomorphism).

The second type has Cayley table given by

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

This group is in fact isomorphic to C_4 (find an explicit isomorphism).

To see that these two groups cannot be isomorphic observe that in the first group, every element (other than the identity) has order 2, but this is not the case in the second group where we have elements of order 4 (for example b).