MA3615 Groups and Symmetry

Solutions to Exercise Sheet 3

1. (a) $H = \{0, 2, 4, 6\}$ is a subgroup of $\mathbb{Z}_8$ of order 4.
   $H' = \{0, 4\}$ is a subgroup of $\mathbb{Z}_8$ of order 2.

   (b) $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$.
   If n is divisible by m, then $n = mk$ for some $k \in \mathbb{N}$.
   Consider $H = \{0, k, 2k, \ldots, (m-1)k\}$. This is a subgroup of $\mathbb{Z}_n$ as we have
   (S1) $0 \in H$.
   (S2) $ik + jk = \begin{cases} (i+j)k \in H & \text{if } i+j \leq m-1 \\ (i+j-m)k \in H & \text{if } i+j > m \end{cases}$
   (S3) $(ik)^{-1} = (m-i)k \in H$ as $ik + (m-i)k = mk = 0$.

   (c) As $\mathbb{Z}_8$ is abelian, left and right cosets coincide. We have

$$\begin{aligned} &H' = \{0, 4\}, \\ &1 + H' = H' + 1 = \{1, 5\}, \\ &2 + H' = H' + 2 = \{2, 6\}, \\ &3 + H' = H' + 3 = \{3, 7\}. \end{aligned}$$

2. $S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$. Any subgroup must have order dividing $|S_3| = 6$, so any subgroup must have order 1,2,3 or 6.
   Order 1: $\{e\}$.
   Order 2: $\{e, (1,2)\}$, $\{e, (1,3)\}$, $\{e, (2,3)\}$.
   Order 3: $\{e, (1,2,3), (1,3,2)\}$.
   Order 6: S_3.

3. (a) $5\mathbb{Z} = \{$all integer multiples of 5$\}$.

   (b) This subgroup is equal to $H = \{4n + 6n \,:\, n, m \in \mathbb{Z}\} \subseteq 2\mathbb{Z}$. But $2 = 4(-1) + 6 \in H$ As 2 generates $2\mathbb{Z}$ we must have $2\mathbb{Z} \subseteq H$. Thus $H = 2\mathbb{Z}$.

   (c) $H = \{2n + 3m \,:\, n, m \in \mathbb{Z}\} \subseteq \mathbb{Z}$. But $1 = 2(-1) + 3 \in H$ and as 1 generates $\mathbb{Z}$ we have $\mathbb{Z} \subseteq H$. Thus $H = \mathbb{Z}$.

4.

$$\begin{aligned} &\langle 1 \rangle = \{1\} \\ &\langle 2 \rangle = \{2, 4, 1\} \\ &\langle 3 \rangle = \{3, 2, 6, 4, 5, 1\} = \mathbb{Z}_7^* \\ &\langle 4 \rangle = \{4, 2, 1\} \\ &\langle 5 \rangle = \{5, 4, 6, 2, 3, 1\} = \mathbb{Z}_7^* \\ &\langle 6 \rangle = \{6, 1\} \end{aligned}$$

   So $\mathbb{Z}_7^*$ is cyclic generated by 3 (or 5).

5. Let s be the reflection through the vertical axis, t be the reflection through the horizontal axis and r be the rotation by π around the centre of the figure. Then $G = \{e, s, t, r\}$. (Note that this group is isomorphic to the group of symmetry of a rectangle, and also isomorphic to the group V_4 introduced in Exercise Sheet 2.)
Now we have $G \subseteq D_8$ and $G \subseteq D_{12}$. In fact we have $G = D_8 \cap D_{12}$ as G has to preserve both the square and the hexagon.

6. Let H, K be subgroups of a group G. We need to show that $H \cap K$ is also a subgroup of G.
(S1) $e \in H$ and $e \in K$ so $e \in H \cap K$.
(S2) If $g_1, g_2 \in H \cap K$ then $g_1, g_2 \in H$ and so $g_1 * g_2 \in H$, also $g_1, g_2 \in K$ so $g_1 * g_2 \in K$. Thus $g_1 * g_2 \in H \cap K$.
(S3) For each $g \in H \cap K$, we have $g^{-1} \in H$ (as $g \in H$) and $g^{-1} \in K$ (as $g \in K$) and so $g^{-1} \in H \cap K$.

7. $D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$ Note that for $1 \leq i \leq 3$ we have $sr^i = r^{4-i}s$.

   (a) $H = \{e, s\}$
   $rH = \{r, rs\}$, $Hr = \{r, r^3s\}$
   $r^2H = \{r^2, r^2s\} = Hr^2$
   $r^3H = \{r^3, r^3s\}$, $Hr^3 = \{r^3, rs\}$.

   (b) $H = \{e, r, r^2, r^3\}$
   $Hs = \{s, r^3s, r^2s, rs\} = Hs$.

8. We have $|S_4| = 4.3.2.1 = 24$, so using Lagrange's theorem we know that any subgroup of S_4 has order 1,2,3,4,6,8,12 or 24. Thus if H is a subgroup with $|H| > 8$ then we know that $|H| \geq 12$.