MA3615 Groups and Symmetry

Solutions to Exercise Sheet 4

1. $H = \{e, (1,2,3), (1,3,2)\}$. We have $|H| = 3$ and $|S_3| = 6$, so $[S_3 : H] = 2$ and hence we know that H is normal in S_3.

   However, as a subgroup of S_4, H is not normal. Take for example $g = (1,4)$, then $g(1,2,3)g^{-1} = (1,4)(1,2,3)(1,4) = (2,3,4) \notin H$.

2. Note that for $\{a, b, c, d\} = \{1, 2, 3, 4\}$ and $g \in S_4$ we have

$$g^{-1}(g(a), g(b))(g(c), g(d))g = (a, b)(c, d)$$

   so we have

$$g(a, b)(c, d)g^{-1} = (g(a), g(b))(g(c), g(d)).$$

   (More generally, for any permutation σ and g we have that $g\sigma g^{-1}$ has the same cycle type as σ.)

   Thus for any $g \in S_4$ we have that $g(a, b)(c, d)g^{-1}$ is a product of two disjoint transpositions. This implies that $g(a, b)(c, d)g^{-1} \in N$ and hence N is normal in S_4.

   Let us now calculate the cosets of N in S_4:
   $N = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.
   $(1,2)N = \{(1,2), (3,4), (1,3,2,4), (1,4,2,3)\}$.
   $(1,3)N = \{(1,3), (1,2,3,4), (2,4), (1,4,3,2)\}$.
   $(2,3)N = \{(2,3), (1,3,4,2), (1,4), (1,2,4,3)\}$.
   $(1,2,3)N = \{(1,2,3), (1,3,4), (2,4,3), (1,4,2)\}$.
   $(1,3,2)N = \{(1,3,2), (1,4,3), (2,3,4), (1,2,4)\}$.

   The Cayley table for S_4/N is given by

	N	$(1,2)N$	$(1,3)N$	$(2,3)N$	$(1,2,3)N$	$(1,3,2)N$
N	N	$(1,2)N$	$(1,3)N$	$(2,3)N$	$(1,2,3)N$	$(1,3,2)N$
$(1,2)N$	$(1,2)N$	N	$(1,3,2)N$	$(1,2,3)N$	$(2,3)N$	$(1,3)N$
$(1,3)N$	$(1,3)N$	$(1,2,3)N$	N	$(1,3,2)N$	$(1,2)N$	$(2,3)N$
$(2,3)N$	$(2,3)N$	$(1,3,2)N$	$(1,2,3)N$	N	$(1,3)N$	$(1,2)N$
$(1,2,3)N$	$(1,2,3)N$	$(1,3)N$	$(2,3)N$	$(1,2)N$	$(1,3,2)N$	N
$(1,3,2)N$	$(1,3,2)N$	$(2,3)N$	$(1,2)N$	$(1,3)N$	N	$(1,2,3)N$

   Note that we can see from the Cayley table that $S_4/N \cong S_3$.

3. As $\mathbb{Z}_8$ is abelian, every subgroup is normal. Take $N = \{0, 4\}$. This is a subgroup (see Exercise sheet 2). Let us compute the cosets:
   $N = \{0, 4\}$.
   $1 + N = \{1, 5\}$.
   $2 + N = \{2, 6\}$.

$3 + N = \{3, 7\}$.
The Cayley table for $\mathbb{Z}_8/N$ is given by

	N	$1+N$	$2+N$	$3+N$
N	N	$1+N$	$2+N$	$3+N$
$1+N$	$1+N$	$2+N$	$3+N$	N
$2+N$	$2+N$	$3+N$	N	$1+N$
$3+N$	$3+N$	N	$1+N$	$2+N$

We can see from the Cayley table that the map $\phi : \mathbb{Z}_8/N \to \mathbb{Z}_4$ given by $a + N \mapsto a$ is an isomorphism.

Alternatively, define $\theta : \mathbb{Z}_8 \to \mathbb{Z}_4$ sending n to its residue modulo 4. Then θ is a homomorphism, and θ is onto as we have $\theta(n) = n$ for $n = 0, 1, 2, 3$. Moreover, $\mathrm{Ker}\theta = \{0, 4\} = N$. Thus from the first isomorphism theorem we get that $\mathbb{Z}_8/N \cong \mathbb{Z}_4$.

4. $G = D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$.
   $N = \{e, r^2\}$.
   $rN = \{r, r^3\} = Nr$.
   $sN = \{s, r^2s\} = Ns$.
   $rsN = \{rs, r^3s\} = Nrs$.
   As left and right cosets coincide we see that N is normal in G.

   Let us compute the Cayley table for G/N:

	N	rN	sN	rsN
N	N	rN	sN	rsN
rN	rN	N	rsN	sN
sN	sN	srN	N	rN
rsN	rsN	sN	rN	N

   Let us compute the Cayley table for $\mathbb{Z}_2 \times \mathbb{Z}_2$:

	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

   Comparing these two Cayley tables we see that the map $\phi : G/N \to \mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\phi(N) = (0, 0)$, $\phi(rN) = (1, 0)$, $\phi(sN) = (0, 1)$ and $\phi(rsN) = (1, 1)$ is an isomorphism.

5. $\theta : \mathbb{Z}_6 \to S_3$ is a homomorphism with $\theta(1) = (1, 2, 3)$. Then we must have
   $\theta(2) = \theta(1 + 1) = \theta(1) \circ \theta(1) = (1, 2, 3)(1, 2, 3) = (1, 3, 2)$.
   $\theta(3) = \theta(2 + 1) = \theta(2) \circ \theta(1) = (1, 3, 2)(1, 2, 3) = e$.
   $\theta(4) = \theta(3 + 1) = \theta(3) \circ \theta(1) = e(1, 2, 3) = (1, 2, 3)$.
   $\theta(5) = \theta(4 + 1) = \theta(4) \circ \theta(1) = (1, 2, 3)(1, 2, 3) = (1, 3, 2)$.
   $\theta(0) = e$.

   Now the kernel of θ is given by $\mathrm{Ker}\theta = \{0, 3\}$ and the image of θ is given by $\mathrm{Im}\theta = \{e, (1, 2, 3), (1, 3, 2)\}$.

6. We need to check that $\phi(\sigma \circ \tau) = \phi(\sigma) + \phi(\tau)$ for all $\sigma, \tau \in S_n$.

   If σ and τ are both odd permutations, then $\sigma \circ \tau$ is an even permutation and we get $\phi(\sigma \circ \tau) = 0$ and $\phi(\sigma) + \phi(\tau) = 1 + 1 = 0$ (modulo 2) as required.

   If precisely one of σ and τ are odd then $\sigma \circ \tau$ is odd and we get $\phi(\sigma \circ \tau) = 1$ and $\phi(\sigma) + \phi(\tau) = 1 + 0 = 1$ as required.

   If σ and τ are both even permutations then $\sigma \circ \tau$ is even and we get $\phi(\sigma \circ \tau) = 0$ and $\phi(\sigma) + \phi(\tau) = 0 + 0 = 0$ as required.

   Now the kernel of ϕ is given by $\text{Ker}\phi = A_n$. As ϕ is onto we get from the first isomorphism theorem that $S_n/A_n \cong \mathbb{Z}_2$.