MA3615 Groups and Symmetry

Solution to Exercise Sheet 5

1. The group G contains the following elements:
$e, g = (1,2,3)(4,5), g^2 = (1,3,2), g^3 = (4,5), g^4 = (1,2,3), g^5 = (1,3,2)(4,5)$
$h = (7,8), gh = (1,2,3)(4,5)(7,8), g^2h = (1,3,2)(7,8), g^3h = (4,5)(7,8)$
$g^4h = (1,2,3)(7,8), g^5h = (1,3,2)(4,5)(7,8)$.
So $|G| = 12$.

   $Orb_G(1) = Orb_G(2) = Orb_G(3) = \{1,2,3\}$ and $G_1 = G_2 = G_3 = \{e, h, g^3, g^3h\}$, so for $i = 1,2,3$ we get
$$|Orb_G(i)|.|G_i| = 3 \times 4 = 12 = |G|.$$

   $Orb_G(4) = Orb_G(5) = \{4,5\}$ and $G_4 = G_5 = \{e, h, g^2, g^4, g^2h, g^4h\}$, so for $j = 4,5$ we get
$$|Orb_G(j)|.|G_j| = 2 \times 6 = 12 = |G|.$$

   $Orb_G(6) = \{6\}$ and $G_6 = G$ so we get
$$|Orb_G(6)|.|G_6| = 1 \times 12 = 12 = |G|.$$

   $Orb_G(7) = Orb_G(8) = \{7,8\}$ and $G_7 = G_8 = \{e, g, g^2, g^3, g^4, g^5\}$, so for $k = 7,8$ we get
$$|Orb_G(k)|.|G_k| = 2 \times 6 = 12 = |G|.$$

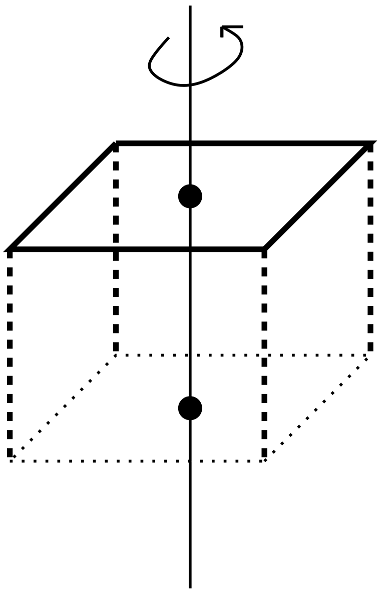
2. Let v be a vertex of the icosahedron. Then $|G| = |Orb_G(v)|.|G_v|$. The vertex v can be mapped to any other vertex by a rotation of the icosahedron. So $|Orb_G(v)| = 12$. At each vertex meet 5 triangular faces. So the stabilizer subgroup of v consists of 5 rotations (including e) around the axis passing through v. Thus $|G_v| = 5$ and we get $|G| = 12 \times 5 = 60$.

3. $D_{12} = \{e, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$.

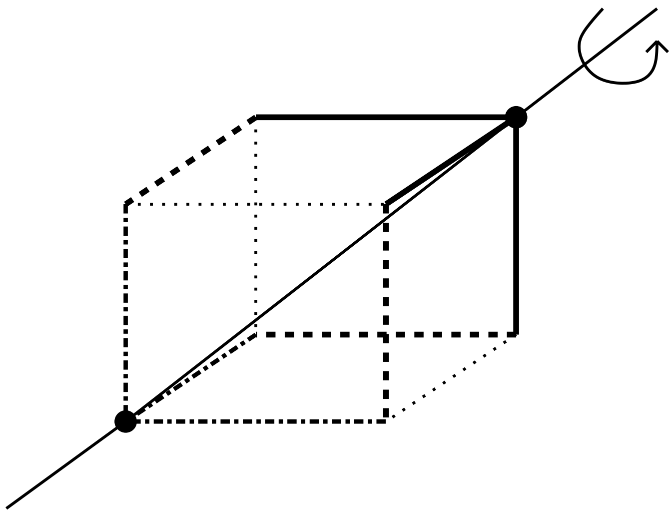
   (a) This action of D_{12} is faithful as no element in D_{12}, other then e, fixes all the sides of the hexagon. The stabilizer of a side consists of e and a reflection across a line passing through the middle of that side and the middle of the opposite side.

   (b) This action of D_{12} is not faithful as the rotation r^3 by π fixes all diagonals. The stabilizer of a diagonal consists of e, r^3, the reflection across that diagonal and the reflection across a line perpendicular to that diagonal.

4. If H is the subgroup generated by a rotation by $\frac{\pi}{2}$ around an axis through the centre of opposite faces. Then we get 3 orbits, each containing 4 edges.



If H is the subgroup generated by a rotation around one of the main diagonal the cube then we get 4 orbits, each containing 3 edges.



5. Let $G = D_{20}$ be the symmetry group of a regular 10-gon. Then $|G| = 20$. The total number of necklaces is given by $\begin{pmatrix} 10 \\ 3 \end{pmatrix} = 120$. Now we have $Fix(e) = 120$. The group G contains 9 rotations and none of them fixes any necklace. It contains 5 reflections across axes passing through two opposite vertices (beads) and each of these fixes precisely 8 necklaces. And finally, it contains 5 more reflections across axes passing through the middle of opposite edges which do not fix any necklace. Using

Burnside Counting theorem we get that the number of distinguishable necklaces is given by

$$\frac{1}{20}(120 + (5 \times 8)) = 8.$$

6. $G = D_8 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$ the symmetry group of the square. So $|G| = 8$. The total number of ID cards is equal to $\begin{pmatrix} 9 \\ 2 \end{pmatrix} = 36$. Now we have $Fix(e) = 36$, $Fix(r) = 0$, $Fix(r^2) = 4$, $Fix(r^3) = 0$, $Fix(s) = 6$, $Fix(rs) = 6$, $Fix(r^2s) = 6$, $Fix(r^3s) = 6$. Thus using Burnside Counting theorem we get that the number of distinguishable ID cards is equal to

$$\frac{1}{8}(36 + 4 + 6 + 6 + 6 + 6) = 8.$$