Solutions to MA3615 Groups and Symmetry May 2011 exam

1. (a) i. G is not a group as the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse.

   ii. H is a group. The multiplication is closed as $1 \times 1 = 1 \in H$, $1 \times 2 = 2 \times 1 = 2 \in H$ and $2 \times 2 = 1 \in H$. Now we have
   (G1) identity element is 1.
   (G2) inverses are given by $1^{-1} = 1$ and $2^{-1} = 2$.
   (G3) associativity follows from associativity of multiplication of integers.

   iii. K is not a group as $(1,2) \circ (1,3) = (1,3,2) \notin K$.

   [6]

   (b) Let $G = \{e, a, b\}$ be a group of order 3 with identity element e. Then its Cayley table starts as

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & & \\ b & b & & \end{array}$$

   Now either $a^2 = e$ or $a^2 = b$. But if $a^2 = e$ then we must have $a * b = b$ and so, by existence of b^{-1}, a would have to be the identity which is a contradiction. Thus we must have $a^2 = b$. Once this is established, there is only one way to complete the rest of the Cayley table and we get

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array}$$

   Thus, up to isomorphism, there is only one possible group of order 3. (In fact it is easy to see that $G \cong C_3$ the cyclic group of order 3). **[8]**

   (c) i. $\psi : \mathbb{Z}_3 \to C_3$ given by $\psi(0) = e$, $\psi(1) = r$ and $\psi(2) = r^2$.
   $\phi : \mathbb{Z}_3 \to C_3$ given by $\phi(0) = e$, $\phi(1) = r^2$ and $\phi(2) = r$.

   ii. $\theta : \mathbb{Z}_3 \to C_3$ given by $\theta(0) = \theta(1) = \theta(2) = e$.

   [3]

   (d) Let D_6 be the group of all symmetries of an equilateral triangle. The group $\mathbb{Z}_6$ is abelian but the groups D_6 and S_3 are not. So $\mathbb{Z}_6$ cannot be isomorphic to D_6 or S_3. Now by labelling the vertices of the triangle by 1,2 and 3 we obtain a natural homomorphism $\phi : D_6 \to S_3$. We need to show that ϕ is a bijection. Note that $|D_6| = |S_3| = 6$ so it is enough to show that ϕ is one-to-one. If $\phi(g) = e$ for some $g \in D_6$ then g fixes every vertex of the triangle. But this implies that g is the identity. So we have that $\mathrm{Ker}\phi = \{e\}$ and hence ϕ is one-to-one as required. **[8]**

2. (a) A subgroup H of a group G is a normal subgroup if $gH = Hg$ for all $g \in G$. **[2]**

   (b) The subgroup H will have exactly two left/right cosets in G, namely H and $\{g \in G : g \notin H\} = C$. This implies that $gH = Hg = C$ for all $g \notin H$. **[4]**

(c) i. $H_1 = \langle rs \rangle = \{e, rs\}$.
Left cosets are given by $H_1 = \{e, rs\}$, $rH_1 = \{r, r^2s\}$, $r^2H_1 = \{r^2, r^3s\}$, $r^3H_1 = \{r^3, s\}$.
Right cosets are given by $H_1 = \{e, rs\}$, $H_1r = \{r, rsr = s\}$, $H_1r^2 = \{r^2, rsr^2 = r^3s\}$, $h_1r^3 = \{r^3, rsr^3 = r^2s\}$. **[3]**

ii. $H_2 = \langle r^2 \rangle = \{e, r^2\}$.
Left/right cosets are given by $H_2 = \{e, r^2\}$, $rH_2 = \{r, r^3\} = H_2r$, $sH_2 = \{s, sr^2 = r^2s\} = H_2s$, $rsH_2 = \{rs, rsr^2 = r^3s\} = H_2rs$.

[3]

As left and right cosets coincide we see that H_2 is normal in D_8.
Now the Cayley table of D_8/H_2 is given by

	H_2	rH_2	sH_2	rsH_2
H_2	H_2	rH_2	sH_2	rsH_2
rH_2	rH_2	H_2	rsH_2	sH_2
sH_2	sH_2	rsH_2	H_2	rH_2
rsH_2	rsH_2	sH_2	rH_2	H_2

Consider the Cayley table for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

Take for example $\theta : D_8/H_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ defined by $\theta(H_2) = (0,0)$, $\theta(rH_2) = (0,1)$, $\theta(sH_2) = (1,0)$ and $\theta(rsH_2) = (1,1)$ then we see from the Cayley tables that θ is an isomorphism. **[6]**

(d) Consider the action of D_8 on the two diagonals of the square (labelled 1 and 2). This gives a surjective homomorphism

$$\phi : D_8 \to S_2$$

given by $\phi(e) = \phi(r^2) = \phi(rs) = \phi(r^3s) = e$ and $\phi(r) = \phi(r^3) = \phi(s) = \phi(r^2s) = (1,2)$.
Hence we have that $N = \text{Ker}\phi = \{e, r^2, rs, r^3s\}$ is a normal subgroup and we can deduce from the First Isomorphism Theorem that $D_8/N \cong S_2$. **[7]**

3. (a) There are 8 rotations (r_1) by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around axes through a vertex and the centre of the opposite face. There are 3 rotations (r_2) by π around axes through the middle of opposite edges. Adding the identity, this gives $|G| = 8 + 3 + 1 = 12$. **[4]**

(b) G' is a subgroup of G. We have $|G'| = 3$ by colouring 3 vertices in one colour and the forth one with a different colour. In this case we have $G' \cong C_3$.
It is not possible to have $|G'| = 5$. By Langrange's theorem we must have that $|G'|$ divides $|G| = 12$, and 5 does not divide 12. **[6]**

(c) Let G be a finite group acting on a finite set X. For $g \in G$, define $\mathrm{Fix}(g)$ to be

$$\mathrm{Fix}(g) = |\{x \in X \mid g(x) = x\}|.$$

Then the number of G-orbits on X is given by

$$\frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g).$$

[3]

(d) Let X be the set of all painted tetrahedrons. Then $|X| = 3^4$ (3 choices of colours for each of the four vertices). Consider the action of G on X. Then the number of different coloured tetrahedrons is equal to the number of G-orbits on X. **[2]**
Now we have
$\mathrm{Fix}(e) = 3^4$
$\mathrm{Fix}(r_1) = 3^2$ (3 vertices painted in one colour, and the forth painted with any colour)
$\mathrm{Fix}(r_2) = 3^2$ (2 pairs of vertices of the same colour)
Thus using Burnside counting theorem we get that the number of different coloured tetrahedrons is given by

$$\frac{1}{12}(3^4 + (8 \times 3^2) + (3 \times 3^2)) = 15.$$

[5]

Description of all different coloured tetrahedrons:
Using only one colour, we can contruct 3 different tetrahedrons.
Using two different colours, there are two possibilities. We can either colour three vertices in one colour and one in another, this gives $3 \times 2 = 6$ different tetrahedrons. Or we can paint two vertices with one colour and the other two vertices in another colour, this gives $\frac{3 \times 2}{2}$ different tetrahedrons.
Using the three colours, we have to colour two vertices with one colour and the other two vertices with the two remaining colours, this gives 3 different tetrahedrons.
Thus we have $3 + 6 + 3 + 3 = 15$ different tetrahedrons. **[5]**

4. (a) We say that the group G acts on the set X if we have a homomorphism

$$\phi : G \to \mathrm{Sym}(X).$$

The G-orbit of x is given by

$$\mathrm{Orb}_G(x) = \{y \in X \ : \ y = g(x) \text{ for some } g \in G\}.$$

The stabilizer G_x of x in G is defined by

$$G_x = \{g \in G \ : \ g(x) = x\}.$$

[6]

(b) (S1) $e \in G_x$ as $e(x) = x$.
(S2) If $g, h \in G_x$, that is $g(x) = h(x) = x$, then $(gh)(x) = g(h(x)) = g(x) = x$ so $gh \in G_x$.
(S3) If $g \in G_x$, that is $g(x) = x$, then $g^{-1}(x) = g^{-1}(g(x)) = (g^{-1}g)(x) = e(x) = x$ so $g^{-1} \in G_x$. **[6]**

(c) For any $x \in X$ we have $|G| = |\mathrm{Orb}_G(x)| \times |G_x|$. **[2]**

(d) i. The group G acts on the set X of dotted edges. Let $x \in X$, then we have $|\mathrm{Orb}_G(x)| = 2$ as there is a rotation mapping x to the other dotted edge. Now G_x consists of e and the rotation by π around the axis through the middle of the dotted edges. Using the Orbit-Stabilizer theorem we get $|G| = 2.2 = 4$. **[4]**

ii. Let G be a finite subgroup of $SO_3(\mathbb{R})$. Then G is isomorphic to one of the following groups: C_n $(n \geq 1)$, D_{2n} $(n \geq 2)$, A_4, S_4, A_5. **[2]**

iii. As $|G| = 4$ we have $G \cong C_4$ or D_4. Now G only has rotations of order 2 or 1 (rotation by π around an axis through the centre of the top and bottom faces, rotations by π around an axis through the middle of the dotted/black edges). However, C_4 has an element of order 4. Thus G cannot be isomorphic to C_4 and hence it must be isomorphic to D_4. **[5]**