MA3615 Groups and Symmetry : Solutions to Class test A

1. [**8 marks**] Consider the following sets with operations.

$$A = \{1, 2, 3\} \quad \text{with multiplication modulo 4,}$$
$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; : \; a, b, c, d \in \mathbb{R} \text{ and } a + b = c + d \right\} \quad \text{with addition of matrices.}$$

Then we have that

- A and B are both groups.
- A is a group but B is not a group.
- ✓ B is a group but A is not a group.
- neither A nor B is a group.

A is not a group as the multiplication is not closed (e.g. $2 \times 2 = 0 \notin A$). B is a group with the identity being the zero matrix, and the inverse of a matrix X being $-X$ (note that adding two matrices in B still gives a matrix in B).

2. [**4 marks**] For each of the following statements, decide whether it is true or false. (Delete as appropriate, no justification required).

   (a) $(5, 7)(6, 7, 5, 2) \in A_7$ True

   (b) $(3, 2, 1)(3, 2, 1) \in A_3$ True

3. [**4 marks**] Let $(G, *)$ be an abelian group, let $g \in G$ be an element of order 5 and let $h \in G$ be an element of order 3. Simplify the following expression as much as possible.

$$\begin{aligned}
[(g * h)^3 * g^9 * h * g^3 * h^5]^{-1} &= (g^{15} * h^9)^{-1} && \text{as } G \text{ is abelian} \\
&= ((g^5)^3 * (h^3)^3)^{-1} \\
&= (e^3 * e^3)^{-1} && \text{as } g \text{ has order 5 and } h \text{ has order 3} \\
&= e^{-1} \\
&= e.
\end{aligned}$$

4. [**8 marks**] Let $G = \{a, b, c\}$. Define two different operations on G, namely $*$ and $\dagger$, by giving the following tables.

$*$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\dagger$	a	b	c
a	a	b	c
b	b	a	b
c	c	b	a

(a) Is $(G, *)$ a group? Yes it is. From the Cayley table we see that c must be the identity. The map $\psi : G \to \mathbb{Z}_3$ given by $\psi(c) = 0$, $\psi(a) = 1$ and $\psi(b) = 2$ is an isomorphism.

(b) Is $(G, \dagger)$ a group? No it is not. The element b appears twice in the second row, so $(G, \dagger)$ cannot be a group.

5. [**6 marks**] Let $G = \{x, y, z, u, v, w\}$ be a group with multiplication $*$ and let $\psi : S_3 \to G$ be an isomorphism satisfying $\psi(e) = z$, $\psi(1, 2) = u$, $\psi(2, 3) = y$ and $\psi(1, 2, 3) = w$.

(a) What is the identity element in G? z

(b) What is u^{-1}? u (as $u^{-1} = \psi(1, 2)^{-1} = \psi((1, 2)^{-1}) = \psi(1, 2) = u$).

(c) What is $u * y$? w (as $u * y = \psi(1, 2) * \psi(2, 3) = \psi((1, 2)(2, 3)) = \psi(1, 2, 3) = w$).