MA3615 Groups and Symmetry : Solutions to Class test B

1. [**8 marks**] Consider the following sets with operations.

$$A = \{1, 2\} \quad \text{with multiplication modulo 3,}$$
$$B = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \quad \text{with addition of matrices.}$$

Then we have that

A and B are both groups.
✓ A is a group but B is not a group.
B is a group but A is not a group.
neither A nor B is a group.

A is a group. If you write down the Cayley table you will see that the multiplication is closed, 1 is the identity and every element is its own inverse. B is not a group as the multiplication is not closed (e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \notin B$).

2. [**4 marks**] For each of the following statement, decide whether it is true or false. (Delete as appropriate, no justification required).

   (a) $(2, 5, 6)(3, 7, 1, 4) \in A_7$ False

   (b) $(1, 4, 2)(4, 2) \in A_4$ False

3. [**4 marks**] Let $(G, *)$ be an abelian group, let $g \in G$ be an element of order 4 and let $h \in G$ be an element of order 5. Simplify the following expression as much as possible.

$$\begin{aligned}
[g^3 * (h * g)^4 * h^3 * g^5 * h^3]^{-1} &= (g^{12} * h^{10})^{-1} && \text{as } G \text{ is abelian} \\
&= ((g^4)^3 * (h^5)^2)^{-1} \\
&= (e * e)^{-1} && \text{as } g \text{ has order 4 and } h \text{ has order 5} \\
&= e^{-1} \\
&= e.
\end{aligned}$$

4. [**8 marks**] Let $G = \{a, b, c\}$. Define two different operations on G, namely $*$ and $\dagger$, by giving the following tables.

$*$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

$\dagger$	a	b	c
a	c	b	a
b	b	c	b
c	a	b	c

(a) Is $(G, *)$ a group? yes it is. From the Cayley table we see that b must be the identity and the map $\psi : G \to \mathbb{Z}_3$ given by $\psi(b) = 0$, $\psi(a) = 1$ and $\psi(c) = 2$ is an isomorphism.

(b) Is $(G, \dagger)$ a group? No it is not. The element b appears twice in the second row so $(G, \dagger)$ cannot be a group.

5. [**6 marks**] Let $G = \{x, y, z, u, v, w\}$ be a group with multiplication $*$ and let $\psi : S_3 \to G$ be an isomorphism satisfying $\psi(e) = x$, $\psi(1, 2) = v$, $\psi(2, 3) = z$ and $\psi(1, 3, 2) = y$.

(a) What is the identity element in G? x.

(b) What is z^{-1}? z (as $z^{-1} = \psi(2, 3)^{-1} = \psi((2, 3)^{-1}) = \psi(2, 3) = z$).

(c) What is $z * v$? y (as $z * v = \psi(2, 3) * \psi(1, 2) = \psi((2, 3)(1, 2)) = \psi(1, 3, 2) = y$).